UNIT - V

$\because |X_T(\omega)|^2$ is a non-negative quantity

i.e $|X_T(\omega)|^2 \geq 0$

E[+ve quantity] = +ve quantity

$E[\geq 0] = \geq 0$

i.e expectation of any positive quantity is always positive. Hence, $S_{XX}(\omega) \geq 0$

Hence proved.

2. If $X(t)$ is real valued function then PSD is also real valued function.

Proof: The PSD of $X(t) = S_{XX}(\omega) = \underset{T\to\infty}{Lt} \dfrac{E[|X_T(\omega)|]^2}{2T}$

If $X(t)$ is a real function then

$\because |X_T(\omega)|^2$ is also a real function.

We know expectation of real valued function ~~$S_{XX}(\omega)$~~ is always real.

i.e. $E[|X_T(\omega)|^2]$ is a real valued function

Hence $S_{XX}(\omega)$ is a real valued function.

3. If $X(t)$ is real valued function then PSD satisfies even symmetry i.e $S_{XX}(\omega) = S_{XX}(-\omega)$

Proof: PSD of $X(t) = S_{XX}(\omega) = F[R_{XX}(\tau)]$

$$= \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{-j\omega\tau}\, d\tau$$

Replace 'ω' by '$-\omega$' on both sides

$$S_{XX}(-\omega) = \int_{-\infty}^{\infty} [R_{XX}(\tau)]\, e^{-j(-\omega)\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} R_{xx}(\tau)\, e^{j\omega\tau}\, d\tau$$

Put $\tau = -t \Rightarrow d\tau = -dt$

$\tau \to -\infty \Rightarrow t = \infty$

$\tau \to \infty \Rightarrow t = -\infty$

$$\therefore S_{xx}(-\omega) = \int_{\infty}^{-\infty} R_{xx}(-t)\, e^{-j\omega t} (-dt)$$

$$= \int_{-\infty}^{\infty} R_{xx}(-t)\, e^{-j\omega t}\, dt$$

$t = \tau$

$dt = d\tau$

$$= \int_{-\infty}^{\infty} R_{xx}(-\tau)\, e^{-j\omega\tau}\, d\tau$$

We know for real values of $X(t)$ ACF satisfies even symmetry i.e., $R_{xx}(\tau) = R_{xx}(-\tau)$

$$\therefore S_{xx}(-\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$= S_{xx}(\omega)$$

$$\therefore S_{xx}(\omega) = S_{xx}(-\omega)$$

Hence the property is proved.

4. Derive expression for power spectrum density or Explain average power of random process.

Sol.: ~~Proof~~:

Statement: The PSD of $x(t) = S_{xx}(\omega) = \underset{T\to\infty}{Lt} \frac{E[|X_T(\omega)|^2]}{2T}$

and

Average power of $X(t) = P_{xx} = R_{xx}(0)$

$$= \underset{T\to\infty}{Lt} \frac{1}{2T} \int_{-\infty}^{\infty} \frac{E[|X_T(\omega)|^2]}{2T}\, d\omega$$

Proof: let $X_T(t)$ is defined by $X_T(t) = X(t)$; $T \le t \le T$

$$F[X_T(t)] = X_T(\omega) = \int_{-\infty}^{\infty} [X_T(t)]\, e^{j\omega t}\, dt$$

$$X_T(\omega) = \int_{-T}^{T} x(t)\, e^{j\omega t}\, dt$$

We know parseval's theorem for energy signals

$$\text{Energy of } g(t) = E = \int_{-\infty}^{\infty} |g(t)|^2 dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |G(\omega)|^2 d\omega$$

$$\text{Energy of } X_T(t) = E = \int_{-\infty}^{\infty} |X_T(t)|^2 dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X_T(\omega)|^2 d\omega$$

$$E = \int_{-T}^{T} |X(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X_T(\omega)|^2 d\omega \rightarrow ①$$

We know relationship b/w energy and average power

$$\text{Average power} = P_{avg} = \underset{T\to\infty}{Lt} \frac{1}{2T} [E]$$

Apply $\underset{T\to\infty}{Lt} \frac{1}{2T} [\ ]$; we get

$$\underset{T\to\infty}{Lt} \frac{1}{2T} \int_{-T}^{T} |X(t)|^2 dt = \underset{T\to\infty}{Lt} \frac{1}{2T} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} |X_T(\omega)|^2 d\omega \right]$$

$$\underset{T\to\infty}{Lt} \frac{1}{2T} \int_{-T}^{T} |X(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underset{T\to\infty}{Lt} \frac{|X_T(\omega)|^2}{2T} d\omega$$

Apply expectations both sides, we get

$$\underset{T\to\infty}{Lt} \frac{1}{2T} \int_{-T}^{T} E[|X(t)|^2]\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underset{T\to\infty}{Lt} \frac{E[|X_T(\omega)|^2]}{2T} d\omega$$

For a WSS R.P $A[E[|X(t)|^2]] = |X(t)|^2$

$$\underset{T\to\infty}{Lt}\frac{1}{2T}\int_{-T} E[|X(t)|^2 dt = \overline{|X(t)|}$$

$$\underset{T\to\infty}{Lt}\frac{1}{2T}\int_{-T}^{T}|X(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}\underset{T\to\infty}{Lt}\frac{E[|X_T(\omega)|^2]}{2T}d\omega$$

We know average power of $X(t) = P_{XX} = R_{XX}(0)$

$$= \underset{T\to\infty}{Lt}\frac{1}{2T}\int_{-T}^{+T}|X(t)|^2 dt$$

Hence second statement is proved.

We know relationship b/w ACF & PSD

The ACF of $X(t) = R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, e^{j\omega\tau}\, d\omega$.

At $\tau = 0$; $R_{XX}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, e^{j\omega(0)}\, d\omega$

$$R_{XX}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, d\omega \rightarrow (2)$$

$S_{XX}(\omega) =$ The PSD of $X(t)$,

$$P_{XX} = R_{XX}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\underset{T\to\infty}{Lt}\frac{E[|X_T(\omega)|^2]}{2T}d\omega \rightarrow (3)$$

Compare (2) and (3), we get

$$\therefore S_{XX}(\omega) = \underset{T\to\infty}{Lt}\frac{E[|X_T(\omega)|^2]}{2T}$$

5. $S_{\dot{X}\dot{X}}(\omega) = \omega^2 S_{XX}(\omega)$

Proof: The PSD of $X(t) = S_{XX}(\omega) = \underset{T\to\infty}{Lt}\frac{E[|X_T(\omega)|^2]}{2T}$

$$= \underset{T\to\infty}{Lt}\frac{E[|F[X_T(t)]|^2]}{2T}$$

Here $X_T(t) \xleftrightarrow{F.T} X_T(\omega) = F[X_T(t)]$

The PSD of $\dot{X}(t) = S_{\dot{X}\dot{X}}(\omega) = \underset{T\to\infty}{Lt}\frac{E[|F[\dot{X}_T(t)]|^2]}{2T}$

We know that $g(t) \xleftrightarrow{F.T} G(\omega)$

$$\frac{d}{dt}(g(t)) \longleftrightarrow j\omega\, G(\omega)$$

$$X_T(t) \longleftrightarrow X_T(\omega)$$

$$\dot{X}_T(t) = \frac{d}{dt}[X_T(t)] \longleftrightarrow j\omega\, X_T(\omega) = F[\dot{X}_T(\omega)]$$

$$S_{\dot{x}\dot{x}}(\omega) = \underset{T\to\infty}{Lt} \frac{E[|j\omega . X_T(\omega)|^2]}{2T}$$

$$= \underset{T\to\infty}{Lt} \frac{E[|j\omega|^2 |X_T(\omega)|^2]}{2T}$$

$$|j\omega|^2 = |j|^2 |\omega|^2$$

$$= 1\cdot(|\omega|^2)$$

$$= \omega^2$$

$$S_{\dot{x}\dot{x}}(\omega) = \underset{T\to\infty}{Lt} \frac{E[\omega^2 |X_T(\omega)|^2]}{2T}$$

$$= \omega^2 \underset{T\to\infty}{Lt} \frac{E[|X_T(\omega)|^2]}{2T}$$

$$\therefore S_{\dot{x}\dot{x}}(\omega) = \omega^2 S_{xx}(\omega)$$

Hence proved.

***6. * Derive relationship between ACF and PSD.
** Derive Weiner-khenchine relation.

Statement: The autocorrelation function(ACF) and power spectrum density (PSD) of random process $X(t)$ forms a fourier transform pair, i.e., $R_{XX}(\tau) \xleftrightarrow{F.T} S_{XX}(\omega)$

1. $S_{XX}(\omega) = F[R_{XX}(\tau)] = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau$

2. $R_{XX}(\tau) = F^{-1}[S_{XX}(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega) e^{j\omega\tau} d\omega$

Proof: The PSD of $X(t)$ is defined by

The PDS of $x(t) = S_{XX}(\omega) = \underset{T\to\infty}{Lt} \frac{E[|X_T(\omega)|^2]}{2T}$

$$\Rightarrow S_{XX}(\omega) = \underset{T\to\infty}{Lt} \frac{E[X_T^*(\omega) X_T(\omega)]}{2T} \quad |x|^2 = x^* \cdot x$$

$$X_T(\omega) = F[X_T(t)] = \int_{-\infty}^{\infty} X_T(t) e^{-j\omega t} dt = \int_{-T}^{T} X(t) e^{-j\omega t} dt$$

Replace t by t_1 $\quad \because x_T(t) = x(t) \; ; -T \le t \le T$

$$\Rightarrow X_T(\omega) = \int_{-T}^{T} x(t_1) e^{-j\omega t_1} dt_1$$

$$X_T^*(\omega) = [X_T(\omega)]^* = \left[\int_{-T}^{T} x(t) e^{-j\omega t} dt\right]^*$$

$$= \int_{-T}^{T} x^*(t) (e^{-j\omega t})^* dt \quad \begin{cases} \because X(t) \text{ is real} \\ x^*(t) = x(t) \\ (e^{-j\theta})^* = e^{j\theta} \end{cases}$$

$$\Rightarrow X_T^*(\omega) = \int_{-T}^{T} x(t) e^{j\omega t} dt.$$

Now $S_{XX}(\omega) = \underset{T\to\infty}{Lt} \frac{E\left[\int_{-T}^{T} x(t) e^{j\omega t} dt \cdot \int_{-T}^{T} x(t_1) e^{-j\omega t_1} d\right.}{2T}$

$$= \underset{T\to\infty}{Lt} \frac{E\left[\int_{-T}^{T}\int_{-T}^{T} [X(t_1)\, x(t)]\, e^{-j\omega(t_1-t)}\, dt_1\, dt\right]}{2T}$$

$$= \underset{T\to\infty}{Lt} \frac{1}{2T} \int_{-T}^{T}\int_{-T}^{T} E\left[x(t_1)\, x(t)\right] dt_1\, dt \; e^{-j\omega(t_1-t)}$$

We know $R_{xx}(t, t_1) = E\left[x(t)\, x(t_1)\right]$

$$\Rightarrow S_{xx}(\omega) = \underset{T\to\infty}{Lt} \frac{1}{2T} \int_{-T}^{T}\int_{-T}^{T} R_{xx}(t_1 t_1)\, e^{-j\omega(t_1-t)}\, dt_1\, dt$$

Applying inverse FT on both sides, we get

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} (S_{xx}(\omega))\, e^{j\omega\tau}\, d\omega = \frac{1}{2\pi}\left[\int_{-\infty}^{\infty} \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{xx}(t_1 t_1)\, e^{-j\omega(t_1-t)}\, dt_1\, dt \cdot e^{j\omega\tau}\, d\omega\right]$$

$$= \underset{T\to\infty}{Lt} \frac{1}{2T} \int_{-T}^{T}\int_{-T}^{T} R_{xx}(t_1\, t_1)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} 1 \cdot e^{-j\omega(t_1-t-\tau)}\, d\omega\right] dt_1\, dt$$

We know $1 \overset{FT}{\longleftrightarrow} 2\pi \cdot \delta(\omega)$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} 1 \cdot e^{-j\omega t}\, d\omega = \delta(t)$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} 1 \cdot e^{-j\omega(t_1-t-\tau)}\, d\omega = \delta(t_1-t-\tau)$$

$$\Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)\, e^{j\omega\tau}\, d\omega = \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{xx}(t, t_1)\, \delta(t_1-t-\tau)\, dt_1\, dt$$

$$= \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T}\left[\int_{-T}^{T} R_{xx}(t_1, t_1)\, \delta(t_1-t-\tau)\, dt_1\right] dt$$

We know properties of impulse function

$$\int_{-\infty}^{\infty} f(t)\, \delta(t)\, dt = f(t)\, \delta(t) \Big|_{t=0}$$

$$= f(0)\, \delta(0)$$

$$= f(0) \times 1$$

$$= f(0)$$

$$\int_{-\infty}^{\infty} f(t)\, \delta(t-t_1) = f(t_1) = f(t)\Big|_{t=t_1}$$

$$\int_{-T}^{T} R_{xx}(t, t_1)\, \delta(t_1 - t - \tau)\, dt_1 = R_{xx}(t, t+\tau) = R_{xx}(t, t_1)\Big|_{t_1 = t}$$

$$t_1 - t - \tau = 0$$

$$t_1 = t + \tau$$

$$\Rightarrow \quad = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} R_{xx}(t, t+\tau)\, dt$$

For WSS random process $\Rightarrow A[R_{xx}(t, t+\tau)]$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} R_{xx}(t, t+\tau)\, dt = R_{xx}(\tau)$$

$$\boxed{\frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega)\, e^{j\omega\tau}\, d\omega = R_{xx}(\tau)} \rightarrow (1)$$

i.e., $R_{xx}(\tau) = I.F.T\,[S_{xx}(\omega)]$

Apply F.T on both sides, we get

$$F[R_{xx}(\tau)] = F\left[I.F.T[S_{xx}(\omega)]\right]$$

$$S_{xx}(\omega) = F[R_{xx}(\tau)] = \int_{-\infty}^{\infty} R_{xx}(\tau)\, e^{-j\omega\tau}\, d\tau \rightarrow (2)$$

Here (1) and (2) equations are also known as Weiner-Khinchine relation.

*Problems:

1. Find powerspectrum density, average power & plot PSD spectrum for the following ACF.

(i) $R_{xx}(\tau) = A \cos\omega_0 t$; $\omega_0 = \frac{2\pi}{T}$ rad/sec

Sol: The PSD of $X(t) = S_{xx}(\omega) = F[R_{xx}(\tau)] = \int_{-\infty}^{\infty} R_{xx}(\tau)\, e^{-j\omega\tau} d\tau$

$$R_{xx}(\tau) = A \cos\omega_0 t$$

$$R_{xx}(\tau) = A\left(\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right)$$

$$= \frac{A}{2} e^{j\omega_0 t} + \frac{A}{2} e^{-j\omega_0 t}.$$

Apply fourier transform, we get

$$F[R_{xx}(\tau)] = F\left[\frac{A}{2} e^{j\omega_0 t} + \frac{A}{2} e^{-j\omega_0 t}\right]$$

By linearity property of FT, we get

$$F[R_{xx}(\tau)] = \frac{A}{2} F[e^{j\omega_0 t}] + \frac{A}{2} F[e^{-j\omega_0 t}]$$

We know $1 \xleftrightarrow{F.T} 2\pi\delta(\omega)$

$$e^{j\omega_0 t} \longleftrightarrow 2\pi\delta(\omega-\omega_0)$$

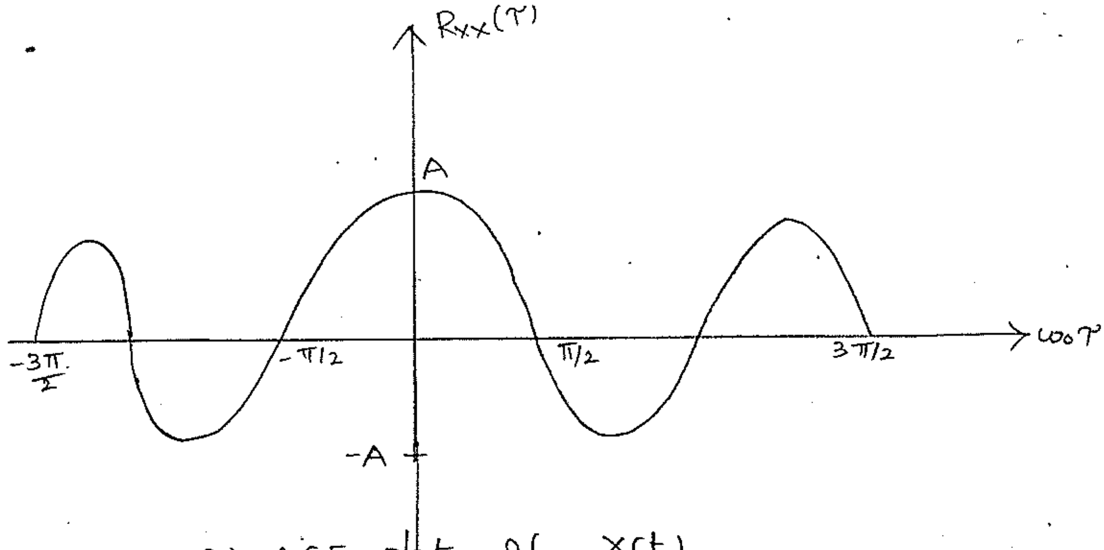
$$\& \; e^{-j\omega_0 t} \longleftrightarrow 2\pi\delta(\omega+\omega_0)$$

$$S_{xx}(\omega) = \frac{A}{2} 2\pi\, \delta(\omega-\omega_0) + \frac{A}{2} 2\pi\,\delta(\omega+\omega_0)$$

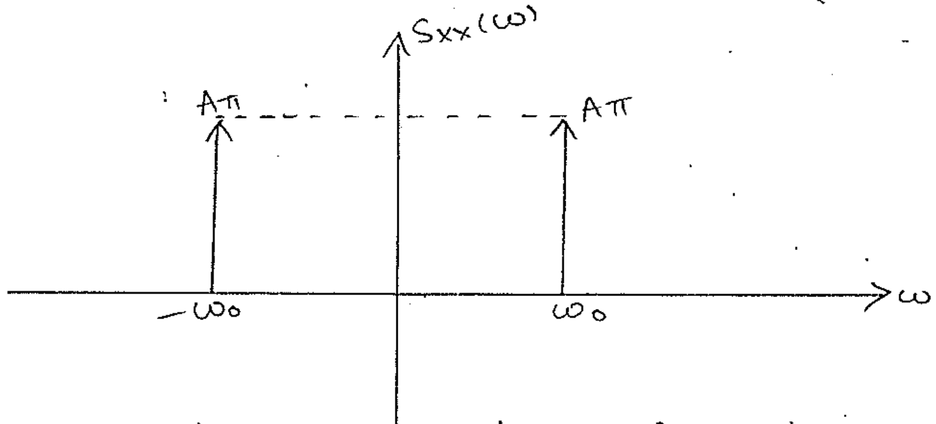
$$= A\pi[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]$$

$\therefore R_{xx}(\tau) = A\cos(\omega_0\tau) \xleftrightarrow{F.T} S_{xx}(\omega) = A\pi[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]$

$$\delta(\omega-\omega_0) = \begin{cases} 1 \; ; & \omega-\omega_0 = 0 \\ & \omega = \omega_0 \\ 0 \; ; & \omega-\omega_0 \neq 0 \\ & \omega \neq \omega_0 \end{cases}$$



(a) ACF plot of $X(t)$



(b) PSD spectrum of $X(t)$

$$\text{Average power} = P_{xx} = R_{xx}(0) = A\cos(\omega_0(0))$$
$$= A\cos 0^\circ$$
$$= A \text{ Watts}$$

(ii) $R_{xx}(t) = A\sin\omega_0 t$; $\omega_0 = \frac{2\pi}{T}$ rad/sec

The PSD of $X(t) = S_{xx}(\omega) = F[R_{xx}(t)] = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-j\omega\tau} d\tau$

$$R_{xx}(t) = A\sin\omega_0 t$$
$$= A\left(\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}\right)$$
$$R_{xx}(t) = \frac{A}{2j} e^{j\omega_0 t} - \frac{A}{2} e^{-j\omega_0 t}$$

Apply fourier transform, we get

$$F[R_{xx}(\cancel{t})] = F\left[\frac{A}{2j} e^{j\omega_0 t} - \frac{A}{2j} e^{-j\omega_0 t}\right]$$

By linearity property of F $\cancel{F}$ we get

$$F[R_{xx}(t)] = \frac{A}{2j} F[e^{j\omega_0 t}] - \frac{A}{2j} F[e^{-j\omega_0 t}]$$

We know $1 \xleftrightarrow{F.T} 2\pi\,\delta(\omega)$

$$e^{j\omega_0 t} \longleftrightarrow 2\pi\,\delta(\omega-\omega_0)$$

$$\& \quad e^{-j\omega_0 t} \longleftrightarrow 2\pi\,\delta(\omega+\omega_0)$$

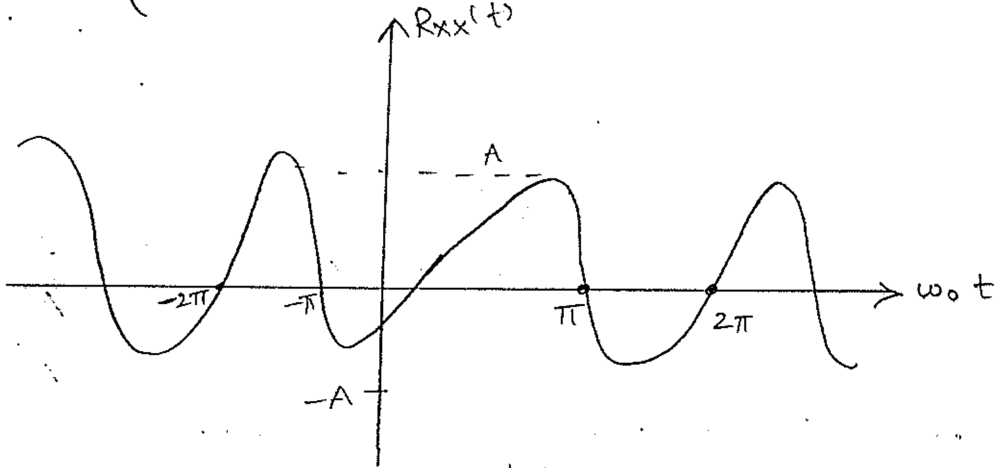
$$S_{XX}(\omega) = \frac{A}{2j}\cdot 2\pi\,\delta(\omega-\omega_0) - \frac{A}{2j}\,2\pi\,\delta(\omega+\omega_0)$$

$$= \frac{A\pi}{j}\left[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right]$$

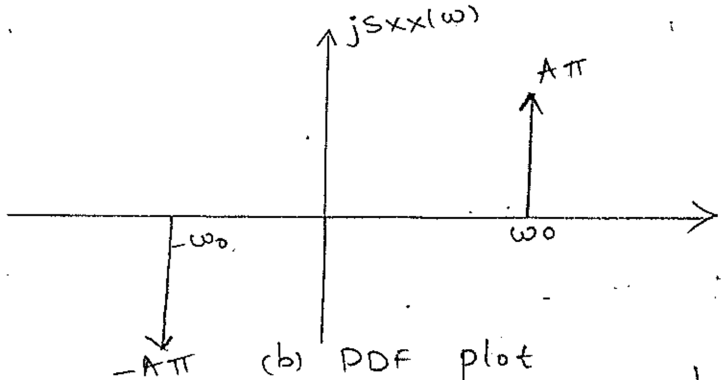
$$\therefore R_{XX}(t) = A\sin(\omega_0 t) \xleftrightarrow{F.T.} S_{XX}(\omega) = \frac{A\pi}{j}\left[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right]$$

$$\delta(\omega-\omega_0) = \begin{cases} 1 ; & \omega-\omega_0 = 0 \Rightarrow \omega=\omega_0 \\ 0 ; & \omega-\omega_0 \neq 0 \Rightarrow \omega \neq \omega_0 \end{cases}$$

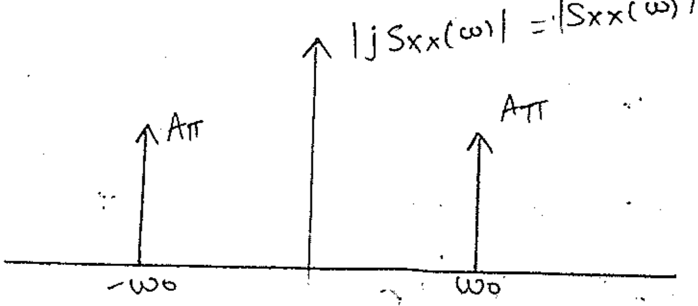
$$\delta(\omega+\omega_0) = \begin{cases} 1 ; & \omega = -\omega_0 \\ 0 ; & \omega \neq -\omega_0 \end{cases}$$



(a) ACF plot



(b) PDF plot

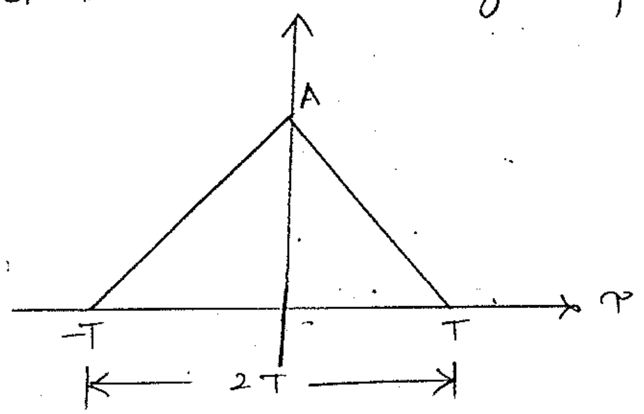


(c) Magnitude spectrum of PSD.

$$\text{Average power} = P_{XX} = K_{XX}(0)$$
$$= A\sin(\omega_0(0))$$
$$\therefore P_{XX} = 0 \text{ Watts}$$

2. Find PSD of for the given autocorrelation function is a basic triangular pulse.

Sol: The given ACF is a basic triangular pulse is as shown in fig



$$(-T, 0) \longrightarrow (0, A) \quad ; (-T \le \tau \le 0)$$
$$x_1 \; y_1 \qquad x_2 \; y_2$$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{A - 0}{0 - (-T)} = \frac{A}{T}$$

$$y = mx + c$$

$$y = x\left(\frac{A}{T}\right) + c$$

$$(0, A) \Rightarrow A = \frac{A}{T}(0) + c$$

$$\therefore c = A$$

$$y = \frac{A}{T}x + A$$

$$y = A\left(1 + \frac{x}{T}\right) \; ; \; -T \le \tau \le 0$$

$$x \to \tau \;\; \& \;\; y \to R_{XX}(\tau)$$

$$\boxed{R_{XX}(\tau) = A\left[1 + \frac{\tau}{T}\right] \; ; \; -T \le \tau \le 0}$$

$(0, A) \longrightarrow (T, 0)$; $\quad 0 \leq \tau \leq T$

$$m = -\frac{A}{T}$$

$$y = -\frac{A}{T} x + c$$

$$(0, A) \Rightarrow c = A$$

$$y = A\left[1 - \frac{x}{T}\right] \quad ; \quad 0 \leq \tau \leq T$$

$$R_{xx}(\tau) = A\left[1 - \frac{\tau}{T}\right] \quad ; \quad 0 \leq \tau \leq T .$$

$$\therefore R_{xx}(\tau) = \begin{cases} A\left(1 + \frac{\tau}{T}\right) & ; \quad -T \leq \tau \leq 0 \\ A\left(1 - \frac{\tau}{T}\right) & ; \quad 0 \leq \tau \leq T \end{cases}$$

(or)

$$R_{xx}(\tau) = A\left[1 - \frac{|\tau|}{T}\right]$$

$$S_{xx}(\omega) = F[R_{xx}(\tau)] = \int_{-\infty}^{\infty} R_{xx}(\tau) \; e^{-j\omega\tau} d\tau$$

$$= \int_{-T}^{0} A\left[1 + \frac{\tau}{T}\right] e^{-j\omega\tau} d\tau + \int_{0}^{T} A\left[1 - \frac{\tau}{T}\right] e^{-j\omega\tau} d\tau$$

$$= A\left(1 + \frac{\tau}{T}\right) \frac{e^{-j\omega\tau}}{-j\omega} - \frac{A}{T} \frac{e^{-j\omega\tau}}{(-j\omega)(-j\omega)}\Bigg]_{-T}^{0} +$$

$$A\left(1 - \frac{\tau}{T}\right) \frac{e^{-j\omega\tau}}{-j\omega} - \left(-\frac{A}{T}\right) \frac{e^{-j\omega\tau}}{(-j\omega)(-j\omega)}\Bigg]_{0}^{T}$$

$$= \left[A\left(\left(1 + \frac{0}{T}\right) \frac{e^{-j\omega\times 0}}{-j\omega} - \frac{A}{T} \frac{e^{-j\omega\times 0}}{(-j\omega)^2}\right] - A\left(1 - \frac{T}{T}\right) \frac{e^{-j\omega(-T)}}{(-j\omega)^2}\right]$$

$$+ A\left(1 - \frac{T}{T}\right) \frac{e^{-j\omega T}}{-j\omega} + \frac{A}{T} \frac{e^{-j\omega T}}{(-j\omega)^2} - \left[A\left(1 - \frac{0}{T}\right) \frac{e^{-j\omega\times 0}}{-j\omega}\right.$$

$$\left. + \frac{A}{T} \frac{e^{-j\omega\times 0}}{(-j\omega)^2}\right]$$

$$= \left(\frac{-A}{j\omega} - \frac{A}{T}\cdot\frac{1}{\omega^2}\right) - \left[1 - \frac{A}{T}\frac{e^{j\omega T}}{-\omega^2}\right] + \left[0 + \frac{A}{T}\frac{e^{-j\omega T}}{-\omega^2}\right]$$

$$- \left[\frac{-A}{j\omega} + \frac{A}{T}\cdot\frac{1}{-\omega^2}\right]$$

$$= \frac{-A}{j\omega} + \frac{A}{T\omega^2} - \frac{A}{T\omega^2} e^{j\omega T} - \frac{A}{T\omega^2} e^{-j\omega T} + \frac{A}{j\omega} + \frac{A}{T\omega^2}$$

$$= \frac{2A}{T\omega^2} - \frac{A}{T\omega^2}\left[e^{j\omega t} + e^{-j\omega T}\right]$$

$$= \frac{2A}{T\omega^2} - \frac{A}{T\omega^2}(2\cos\omega T)$$

$$= \frac{2A}{T\omega^2}(1 - \cos(\omega T))$$

$$= \frac{2A}{T\omega^2}\left[2\sin^2\left(\frac{\omega T}{2}\right)\right]$$

$$= \frac{4A}{T\omega^2}\left[\sin\left(\frac{\omega T}{2}\right)\right]^2$$

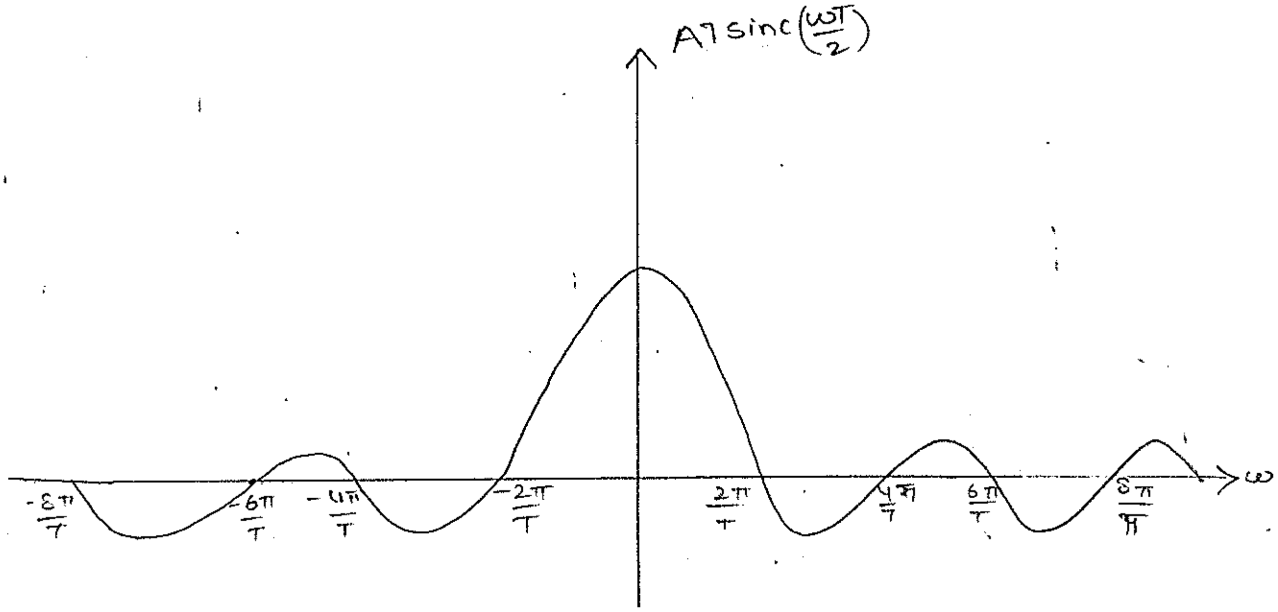
$$= \frac{1}{T}\cdot\frac{4AT}{T\omega^2}\left[\sin\left(\frac{\omega T}{2}\right)\right]^2$$

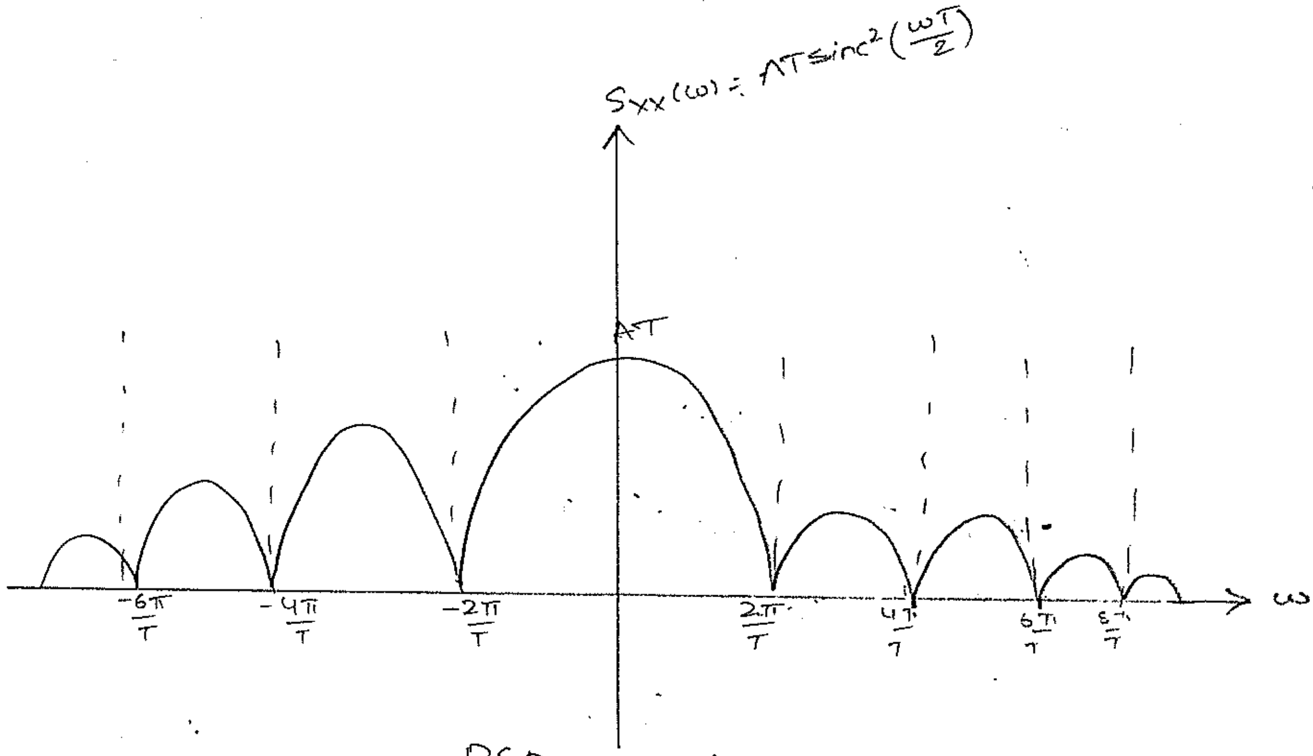
$$= AT\left[\frac{\sin\left(\frac{\omega T}{2}\right)}{\left(\frac{\omega T}{2}\right)}\right]^2$$

$$= AT\left[\text{sinc}\left(\frac{\omega T}{2}\right)\right]^2 \qquad (\because Sa = \text{sampling})$$

$$S_{xx}(\omega) = AT\,\text{sinc}^2\left(\frac{\omega T}{2}\right) = AT\, Sa^2\left(\frac{\omega T}{2}\right)$$

$$\therefore R_{XX}(\tau) = A\,\Delta\left(\frac{\tau}{2T}\right) \xleftrightarrow{F.T.} S_{XX}(\omega) = AT\,\text{sinc}^2\left(\frac{\omega T}{2}\right)$$





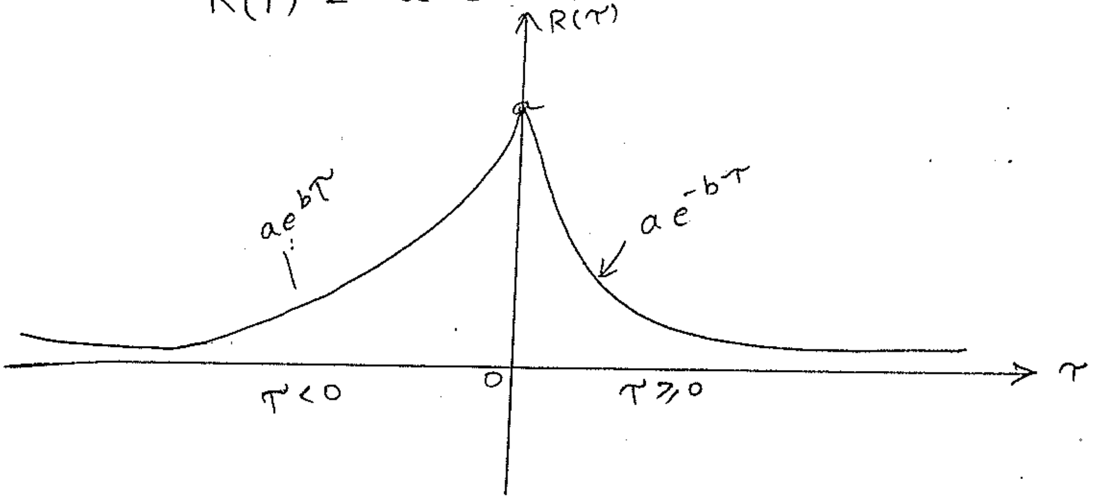
PSD spectrum.

3. Let $R(\tau) = a e^{-b|\tau|}$. Find PSD, average power and plot PSD spectrum.

Sol: (a) Given ACF is $R(\tau) = a e^{-b|\tau|}$

$$R(\tau) = \begin{cases} a e^{b\tau} & ; \tau < 0 \\ a e^{-b\tau} & ; \tau \geq 0 \end{cases} \qquad \because |\tau| = \begin{cases} -\tau & ; \tau < 0 \\ \tau & ; \tau \geq 0 \end{cases}$$

$$R(\tau) = a e^{b\tau} u(-\tau) + a e^{-b\tau} u(\tau)$$



$$F[R(\tau)] = S(\omega) = \int_{-\infty}^{\infty} R(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{0} R(\tau)\, e^{-j\omega\tau} d\tau + \int_{0}^{\infty} R(\tau)\, e^{-j\omega\tau} d\tau$$

$$= \int_{-\infty}^{0} a\,[e^{b\tau}]\, e^{-j\omega\tau} d\tau + \int_{0}^{\infty} a\,[e^{-b\tau}]\, e^{-j\omega\tau} d\tau$$

$$= a \int_{-\infty}^{0} e^{(b-j\omega)\tau} d\tau + a \int_{0}^{\infty} e^{-(b+j\omega)\tau} d\tau$$

$$= a \left[\frac{e^{(b-j\omega)\tau}}{(b-j\omega)} \right]_{-\infty}^{0} + a \left[\frac{e^{-(b+j\omega)\tau}}{-(b+j\omega)} \right]_{0}^{\infty}$$

$$= a \left[\frac{e^{0}}{(b-j\omega)} - 0 \right] + a \left[0 - \frac{e^{0}}{-(b+j\omega)} \right]$$

$$= a \left[\frac{1}{b-j\omega} \right] + a \left[\frac{1}{(b+j\omega)} \right]$$

$$= a\left[\frac{1}{(b+j\omega)} + \frac{1}{(b-j\omega)}\right]$$

$$= a\left(\frac{(b-\cancel{j\omega})+(b+\cancel{j\omega})}{b^2-j^2\omega^2}\right)$$

$$= a\,\frac{2b}{b^2+\omega^2} \qquad \because j^2=-1$$

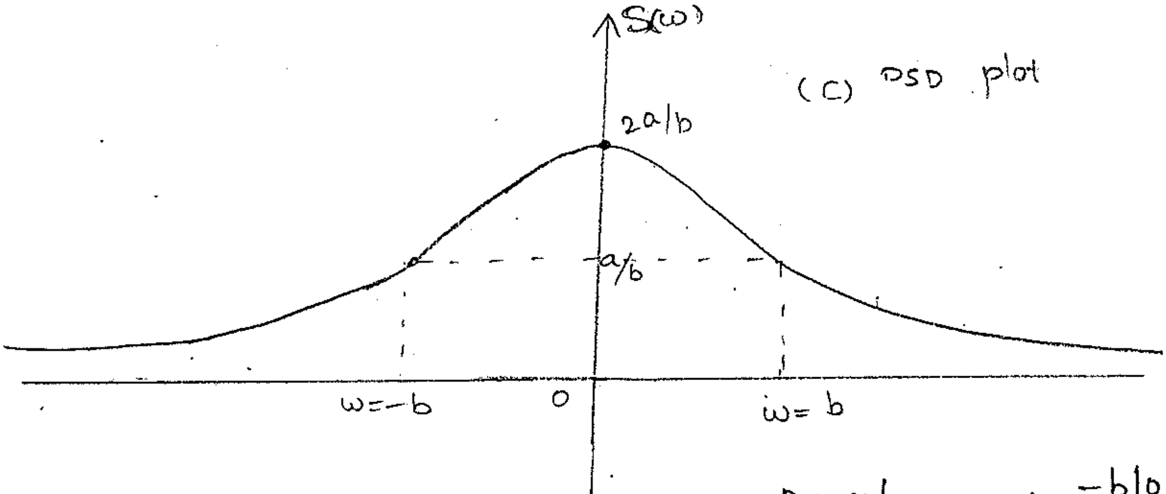
$$F[R(\tau)] = \frac{2ab}{b^2+\omega^2}$$

$$\boxed{\therefore R(\tau) = a e^{-b|\tau|} \xleftrightarrow{F.T} S(\omega) = \frac{2ab}{b^2+\omega^2}}$$

(c) $\omega=0;\ S(0) = \dfrac{2ab}{b^2+0^2} = \dfrac{2ab}{b^2}$

$$S(0) = \frac{2a}{b}$$

$\omega=\pm b;\ S(\pm b) = \dfrac{2ab}{b^2+(\pm b)^2} = \dfrac{a}{b}$



(b) Average power $= P = R(0) = R(\tau)\Big|_{\tau=0} = a e^{-b|0|} = a\times 1$

$$R(0) = a\ (\omega)$$

or

$$P = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2ab}{b^2+\omega^2}\,d\omega$$

$$= \frac{ab}{\pi}\int_{-\infty}^{\infty}\frac{1}{b^2+\omega^2}\,d\omega$$

$$= \frac{a\cancel{b}}{\pi} \cdot \frac{1}{\cancel{b}} \tan^{-1}\left(\frac{\omega}{b}\right)\Big|_{-\infty}^{\infty}$$

$$= \frac{a}{\pi}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right)$$

$$= \frac{a}{\pi} \times \pi$$

$$= a$$

$$\boxed{\therefore\ P = a(\omega)}$$

4. Let a random process $X(t) = A\cos(\omega_0 t + \theta)$ where A, ω_0 are constants and θ is uniformly distributed random variable in the interval $(0, \pi)$, then

(i) the given ~~RSP~~ $X(t)$ is WSS or not

(ii) the average power using time average of second momen of $X(t)$.

(iii) PSD and plot its spectrum.

Sol: Given random process $X(t) = A\cos(\omega_0 t + \theta)$

Here A and ω_0 are constants and θ is uniformly distributed random variable in the interval $(0, \pi)$.

The PDF of 'θ' = $f(\theta) = \begin{cases} \frac{1}{\pi} & ; 0 \le \theta \le \pi \\ 0 & ; \text{otherwise} \end{cases}$

$\because f(x) = \begin{cases} \frac{1}{b-a} & ; a \le x \le b \\ 0 & ; \text{otherwise} \end{cases}$

(i) Condition for WSS random process.

(a) The mean of $X(t) = E[X(t)] =$ constant

(b) The ACF of $X(t) = R_{XX}(t, t+\tau) = E[X(t)\,X(t+\tau)] = R_{XX}(\tau)$

(a) $E[X(t)] = E[A\cos(\omega_0 t + \theta)]$

$$= \int_{-\infty}^{\infty} X(t)\, f_\theta(\theta)\, d\theta$$

$$\neq \int_{0}^{\pi} A = \int_{0}^{\pi} A\cos(\omega_0 t+\theta)\left(\frac{1}{\pi}\right) d\theta$$

$$= \frac{A}{\pi}\left[\frac{\sin(\omega_0 t+\theta)}{1}\right]_0^{\pi}$$

$$= \frac{A}{\pi}\left[\sin(\omega_0 t+\pi) - \sin(\omega_0 t+0)\right]$$

$$= \frac{A}{\pi}\left[-\sin\omega_0 t - \sin\omega_0 t\right] \qquad \boxed{\because \sin(\pi+\theta) = -\sin\theta}$$

$$= \frac{-2A\sin\omega_0 t}{\pi}$$

$\neq$ constant

The ACF of $x(t) = R_{xx}(t, t+\tau) = E[x(t)\, x(t+\tau)]$

$$= \int_{-\infty}^{\infty} x(t)\, x(t+\tau)\, f_\theta(\theta)\, d\theta$$

$$= \int_0^{\pi} A\cos(\omega_0 t+\theta)\, A\cos(\omega_0(t+\tau)+\theta)\left(\frac{1}{\pi}\right) d\theta$$

$$= \frac{A^2}{\pi}\int_0^{\pi} \cos(\omega_0 t+\theta)\cos(\omega_0(t+\tau)+\theta)\, d\theta$$

$$= \frac{A^2}{\pi}\int_0^{\pi} \frac{\cos\omega_0\tau + \cos(2\omega_0 t+\omega_0\tau+2\theta)}{2}\, d\theta$$

$$= \frac{A^2}{2\pi}\cos(\omega_0\tau)\int_0^{\pi} 1 \cdot d\theta + \frac{A^2}{2\pi}\int_0^{\pi}\cos(2\omega_0 t+\omega_0\tau+2\theta)\, d\theta$$

$$= \frac{A^2}{2\pi}\cos(\omega_0\tau)\cdot(\pi-0) + \frac{A^2}{2\pi}\left.\frac{\sin 2\omega_0 t+\omega_0\tau+2\theta}{2}\right|_0^{\pi}$$

$$= \frac{A^2}{2\pi}\cos(\omega_0\tau)(\pi) + \frac{A^2}{4\pi}\left[\sin 2\omega_0 t + \omega_0\tau + 2\times\pi - \sin 2\omega_0 t + \omega_0\tau + 0\right]$$

$$= \frac{A^2 \cos\omega_0\tau}{2} + \frac{A^2}{4\pi}\left[\sin(\cancel{2\omega_0 t} + \omega_0\tau) - \sin(\cancel{2\omega_0 t} + \omega_0\tau)\right]$$

$$= \frac{A^2}{2}\cos\omega_0(\tau) \qquad \boxed{\because \sin(2\pi+\theta) = \sin\theta}$$

$$= R_{xx}(\tau)$$

which is a function of 'τ' only, not absolute time 't'.

Hence the given random process is not a WSS random process since the conditions were not satisfied.

(ii) Average power using time average of 2nd moment of $X(t)$

$$P_{xx} = A\left[E\left[X^2(t)\right]\right]$$

$E[x^2(t)] =$ 2nd moment of $X(t)$

$$= \int_{-\infty}^{\infty} x^2(t)\, f_\theta(\theta)\, d\theta$$

$$= \int_0^{\pi} [A\cos(\omega_0 t + \theta)]^2 \cdot \frac{1}{\pi}\, d\theta$$

$$= \frac{A^2}{\pi}\int_0^{\pi}\left[\frac{1 + \cos 2(\omega_0 t + \theta)}{2}\right] d\theta$$

$$= \frac{A^2}{2\pi}\int_0^{\pi} 1\, d\theta + \frac{A^2}{2\pi}\int_0^{\pi} \cos 2(\omega_0 t + \theta)\, d\theta$$

$$= \frac{A^2}{2\pi}\cdot[\pi - 0] + \frac{A^2}{2\pi}\left[\frac{\sin 2\omega_0 t + 2\theta}{2}\right]_0^{\pi}$$

$$= \frac{A^2}{2} + \frac{A^2}{4\pi}\left[\frac{\sin(2\omega_0 t + 2\pi) - \sin(2\omega_0 t + 0)}{1}\right]$$

$$= \frac{A^2}{2} + \frac{A^2}{4\pi}\left[\sin(2\omega_0 t) - \sin(2\omega_0 t)\right]$$

$$\therefore E[x^2(t)] = \frac{A^2}{2}$$

$$P_{xx} = A\,[E[x^2(t)]]$$

$$= \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T} [E[x^2(t)]]\,dt$$

$$= \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T} \left[\frac{A^2}{2}\right] dt$$

$$= \frac{A^2}{2} \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T} 1 \cdot dt$$

$$= \frac{A^2}{2} \underset{T\to\infty}{Lt} \frac{1}{2T}[T+T]$$

$$= \frac{A^2}{2} \underset{T\to\infty}{Lt} \frac{1}{\cancel{2T}}[\cancel{2T}]$$

$$= \frac{A^2}{2} \underset{T\to\infty}{Lt}\ 1.$$

$$\boxed{P_{xx} = \frac{A^2}{2}\ \text{watts}}$$

(or)

$$P_{xx} = R_{xx}(0) = R_{xx}(\tau)\big|_{\tau=0} = \frac{A^2}{2}\cos\omega_0(\tau)\big|_{\tau=0}$$

$$= \frac{A^2}{2}\cos(\omega_0 \times 0)$$

$$\therefore P_{xx} = \frac{A^2}{2}\ W.$$

(iii) PSD and plot its spectrum.

The ACF of $x(t) = \frac{A^2}{2}\cos\omega_0\tau = R_{xx}(\tau)$

$$F[R_{xx}(\tau)] = F\left[\frac{A^2}{2}(\cos(\omega_0 \tau))\right]$$

We know $\cos\omega_0 t = \dfrac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$

$$= \frac{A^2}{2}\left[\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right]$$

$$R_{xx}(\tau) = \frac{A^2}{4}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right]$$

Applying F.T on both sides

$$F[R_{xx}(\tau)] = F\left[\frac{A^2}{4}\left[e^{j\omega_0 \tau} + e^{-j\omega_0 t}\right]\right]$$

By linearity property, we have

$$= \frac{A^2}{4} F[e^{j\omega_0 t}] + \frac{A^2}{4} F[e^{-j\omega_0 t}]$$

We know $1 \xleftrightarrow{F.T.} 2\pi\,\delta(\omega)$

$$e^{j\omega_0 t} \longleftrightarrow 2\pi\,\delta(\omega-\omega_0)$$

$$e^{-j\omega_0 t} \longleftrightarrow 2\pi\,\delta(\omega+\omega_0)$$

$$S_{xx}(\omega) = \frac{A^2}{4}\cdot 2\pi\,\delta(\omega-\omega_0) + \frac{A^2}{4}\cdot 2\pi\,\delta(\omega+\omega_0)$$

$$S_{xx}(\omega) = \frac{A^2}{2}\pi\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]$$

$$\therefore R_{xx}(\tau) = \frac{A^2}{2}\cos\omega_0 t \xleftrightarrow{F.T.} S_{xx}(\omega) = \frac{A^2}{2}\pi[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)$$

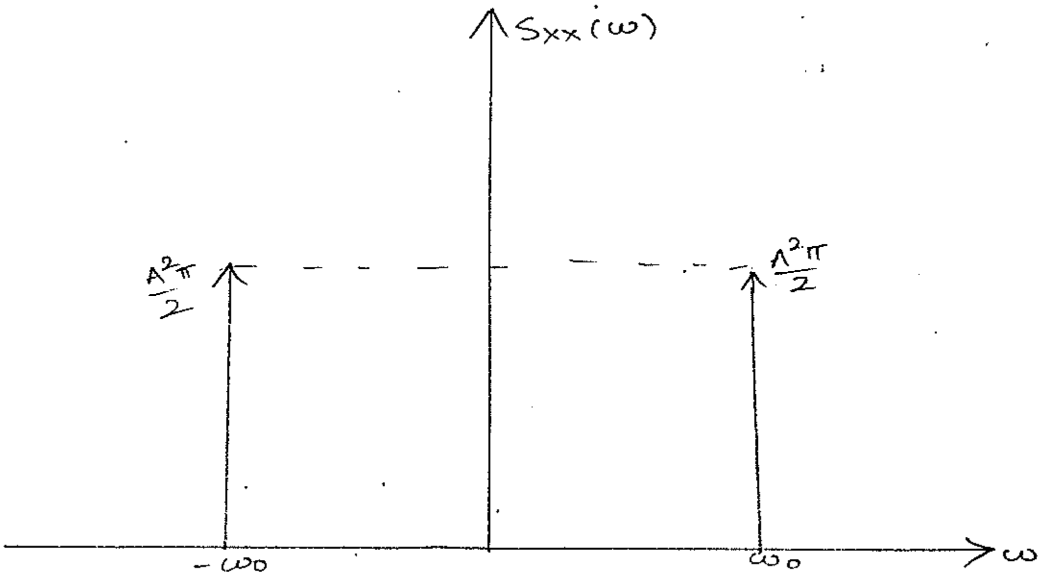


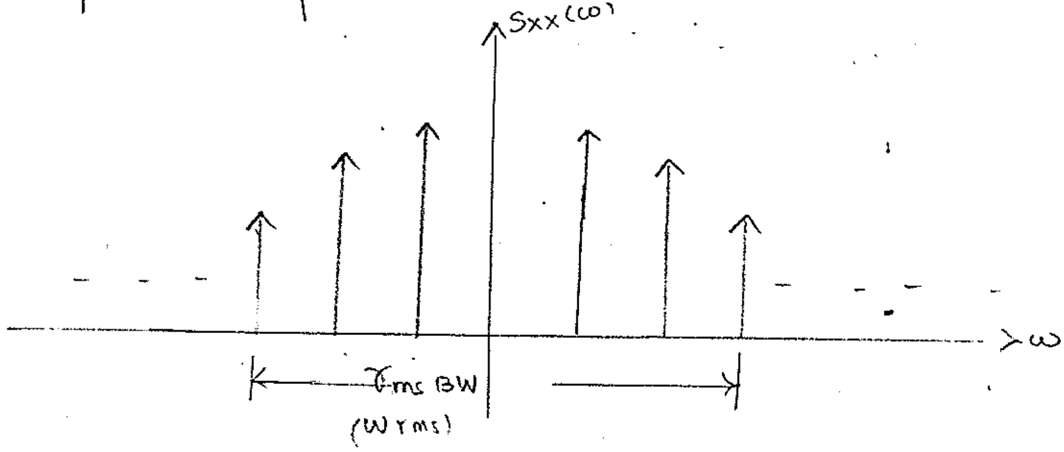
fig: PSD spectrum

*Bandwidth of Power Density Spectrum:

1. Base Band Process:-

RMS Band Width:

The normalisation of a powerspectrum is a measure of its spread it is called RMS bandwidth.

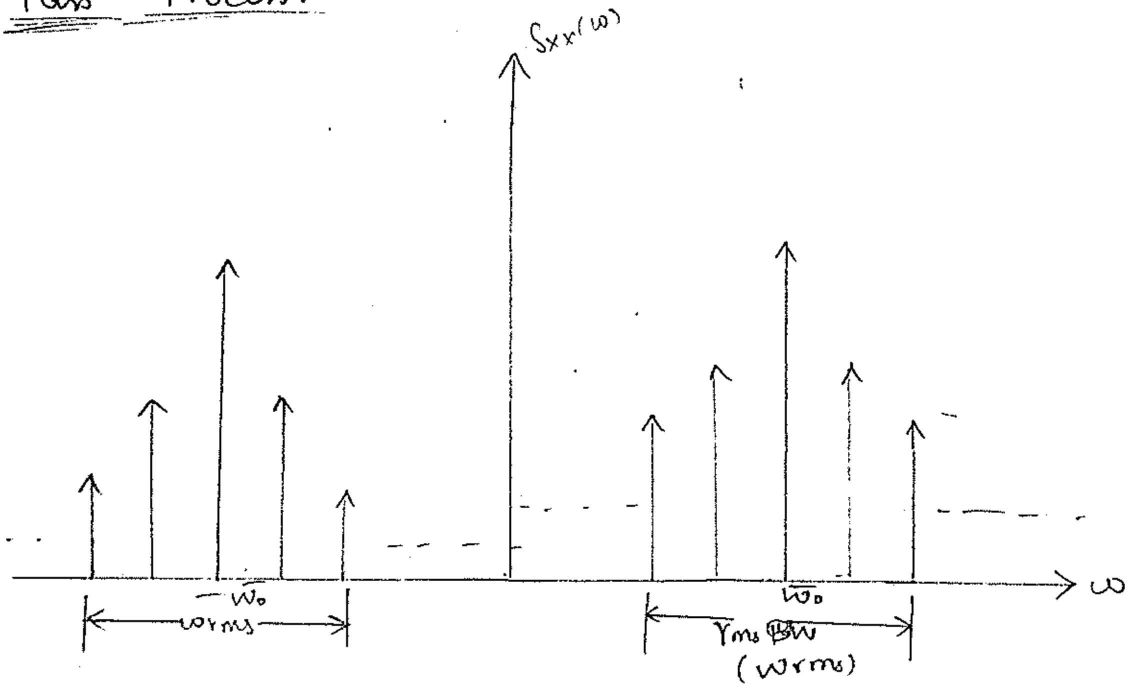


$$\omega^2_{rms} = \frac{\int_{-\infty}^{\infty} \omega^2 S_{XX}(\omega)\, d\omega}{\int_{-\infty}^{\infty} S_{XX}(\omega)\, d\omega}$$

rms BW = ω_{rms} rad/sec

$S_{XX}(\omega)$ — PSD of $X(t)$

Band Pass Process:



$$\overline{\omega_0}^2 = \frac{\int_0^{\infty} \omega^2 S_{XX}(\omega)\, d\omega}{\int_0^{\infty} S_{XX}(\omega)\, d\omega}$$

$$\omega_{rms}^2 = \frac{4 \int_0^{\infty} (\omega - \overline{\omega}_0)^2 S_{XX}(\omega)\, d\omega}{\int_0^{\infty} S_{XX}(\omega)\, d\omega}$$

$\overline{\omega}_0$ = The mean frequency in rad/sec.

In a band pass process, the spectral components are clustered near some frequencies $\overline{\omega}_0$ and $-\overline{\omega}_0$.

*Problems:

1. The PSD of base band random process $X(t)$ is $S_{XX}(\omega) = \frac{2}{\left[1+\left(\frac{\omega}{2}\right)^2\right]^2}$. Find rms band width.

Sol: Given $S_{XX}(\omega) = \frac{2}{\left[1+\left(\frac{\omega}{2}\right)^2\right]^2}$

$$= \frac{2 \times 16}{[4+\omega^2]^2}$$

$$= \frac{32}{(4+\omega^2)^2}$$

We know $\omega_{rms}^2 = \frac{\int_{-\infty}^{\infty} \omega^2 S_{XX}(\omega)\, d\omega}{\int_{-\infty}^{\infty} S_{XX}(\omega)\, d\omega}$

Numerator: $\int_{-\infty}^{\infty} \omega^2 S_{XX}(\omega)\, d\omega = \int_{-\infty}^{\infty} \frac{32\omega^2}{(4+\omega^2)^2}\, d\omega$

$\because \int \frac{x^2}{(a^2+x^2)^2}\, dx = \frac{-x}{2(a^2+x^2)} + \frac{1}{2a}\tan^{-1}\left(\frac{x}{a}\right)$

$$\int_{-\infty}^{\infty} \omega^2 S_{XX}(\omega)\, d\omega = \int_{-6}^{6} \omega^2 \cdot \left(4 - \frac{\omega^2}{9}\right) d\omega$$

$$\left[\int_{-\infty}^{\infty} S_{XX}(\omega) d\omega = 2\pi\, P_{XX} = 2\pi(5.09) = 31.98\right]$$

$$= 4\int_{-6}^{6} \omega^2\, d\omega - \frac{1}{9}\int_{-6}^{6} \omega^4\, d\omega$$

$$= 4\left[\frac{\omega^3}{3}\right]_{-6}^{6} - \frac{1}{9}\left[\frac{\omega^5}{5}\right]_{-6}^{6}$$

$$= 4\left[\frac{216+216}{3}\right] - \frac{1}{45}\left[7776+7776\right]$$

$$= 230.4$$

$$\omega^2_{rms} = \frac{230.4}{31.98} = 7.20$$

rms bandwidth $= \omega_{rms} = \sqrt{7.20} = 2.68$ rad/sec

3. The PSD of $X(t)$ is $S_{XX}(\omega) = \dfrac{6\omega^2}{1+\omega^4}$. Find P_{XX}.

Sol: Given $S_{XX}(\omega) = \dfrac{6\omega^2}{1+\omega^4}$

$$\text{Average Power} = P_{XX} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{6\omega^2}{1+\omega^4}\, d\omega$$

$$= \frac{1}{2\pi}\cdot 6\int_{-\infty}^{\infty} \frac{\omega^2}{1+\omega^4}\, d\omega$$

$$= \frac{3}{\pi}\cdot\frac{\pi}{2\sqrt{2}} \times 1 \qquad \because \int_{-\infty}^{\infty} \frac{x^2}{a+x^4}\, dx = \frac{\pi}{2\sqrt{2}}\, a$$

$$\boxed{\therefore P_{XX} = \frac{3}{2\sqrt{2}} \text{ watts}}$$

4. The PSD of $X(t)$ is $S_{XX}(\omega) = \begin{cases} A\cos\left(\frac{\pi\omega}{2\omega}\right); & |\omega| \le \omega \\ 0; & \text{otherwise}. \end{cases}$

Find P_{XX}?

Sol: Given $S_{XX}(\omega) = \begin{cases} A \cos\left(\frac{\pi\omega}{2\omega}\right); & |\omega| \le \omega \\ 0 & ; \text{otherwise} \end{cases}$

$$\text{Average power} = P_{XX} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega}^{\omega} A\cos\left(\frac{\pi\omega}{2\omega}\right) d\omega$$

$$= \frac{A}{2\pi} \left. \frac{\sin\left(\frac{\pi\omega}{2W}\right)}{\left(\frac{\pi}{2W}\right)} \right|_{-\omega}^{\omega}$$

$$= \frac{A}{2\pi} \times \frac{2\omega}{\pi}\left[\sin\left(\frac{\pi\omega}{2W}\right) - \sin\left(\frac{\pi \times -\omega}{2W}\right)\right]$$

$$= \frac{A\omega}{\pi^2}(2)$$

$$\boxed{\therefore P_{XX} = \frac{2A\omega}{\pi^2} \text{ (watts)}}$$

5. The PSD of random process $X(t)$ is $S_{XX}(\omega) = \begin{cases} A; & |\omega| \le k \\ 0; & \text{otherwise} \end{cases}$

Find (i) the ACF of $X(t)$.

(ii) the average power

(iii) the RMS value.

Sol: Given $S_{XX}(\omega) = \begin{cases} A; & |\omega| \le k \text{ or } -k \le \omega \le k \\ 0; & \text{otherwise} \end{cases}$

$$\text{The ACF of } X(t) = R_{XX}(\tau) = F^{-1}[S_{XX}(\omega)]$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} [S_{XX}(\omega)]\, e^{j\omega\tau}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-k}^{k} A\, e^{j\omega\tau}\, d\omega$$

$$= \frac{A}{2\pi}\left[\frac{e^{j\omega\tau}}{j\tau}\right]_{-k}^{k}$$

$$\times \left[= \frac{A}{2j\pi\tau}\left[e^{j\omega k} - e^{-j\omega k}\right]\right.$$

$$= \frac{A}{\pi\tau}\left(\frac{e^{j\omega k} - e^{-j\omega k}}{2j}\right)$$

$$R_{xx}(\tau) = \frac{A}{\pi\tau}\sin(\omega k)$$

$$= \frac{A\omega k}{\pi\tau}\frac{\sin\omega k}{\omega k}$$

$$= \frac{A\omega k}{\pi\tau}\cdot\operatorname{sinc}(\omega k)$$

$$\left.= A\right] \times$$

$$= \frac{A}{2\pi}\left(\frac{e^{j\omega\tau}}{j\tau}\right)_{-k}^{k}$$

$$= \frac{A}{2\pi j\tau}\left[e^{jk\tau} - e^{-jk\tau}\right]$$

$$= \frac{A}{\pi\tau}\left(\frac{e^{jk\tau} - e^{-jk\tau}}{2j}\right)$$

$$= \frac{A}{\pi\tau}\sin(k\tau)$$

$$= \frac{Ak}{\pi}\frac{\operatorname{sinc}(k\tau)}{(k\tau)} \qquad \because Sa(\theta) = \frac{\sin\theta}{\theta}$$

$$\boxed{\therefore R_{xx}(\tau) = \frac{Ak}{\pi} Sa(k\tau)}$$

ii) Average power $= P_{xx} = R_{xx}(0) = \frac{AK}{\pi} Sa(K \times 0)$

$\because Sa(0) = 1$

$$\therefore P_{xx} = \frac{AK}{\pi} \text{ watts}$$

iii) RMS Band width

$$\omega^2_{rms} = \frac{\int_{-\infty}^{\infty} \omega^2 \cdot S_{xx}(\omega)\, d\omega}{\int_{-\infty}^{\infty} S_{xx}(\omega)\, d\omega}$$

$$P_{xx} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xx}(\omega)\, d\omega \Rightarrow \int_{-\infty}^{\infty} S_{xx}(\omega) d\omega = 2\cancel{\pi} \cdot \frac{AK}{\cancel{\pi}}$$

$$= 2AK$$

$$\Rightarrow \int_{-\infty}^{\infty} \omega^2 S_{xx}(\omega)\, d\omega = \int_{-K}^{K} \omega^2 A\, d\omega$$

$$= A \left[\frac{\omega^3}{3}\right]_{-K}^{K}$$

$$= A \left[\frac{K^3 - (-K^3)}{3}\right]$$

$$= \frac{2AK^3}{3}$$

$$\therefore \omega^2_{rms} = \frac{2AK^3}{3} \Big/ 2AK$$

$$= \frac{\cancel{2}\cancel{A}K^{\cancel{3}\,2}}{3} \times \frac{1}{\cancel{2}\cancel{A}\cancel{K}}$$

$$\omega^2_{rms} = \frac{K^2}{3}$$

rms band width $= \omega_{rms} = \frac{K}{\sqrt{3}}$ rad/sec

6. $S_{XX} = \dfrac{\omega^2}{\omega^4 + 10\omega^2 + 9}$. Find ACF of $X(t)$.

Sol: $$S_{XX} = \frac{\omega^2}{\omega^4 + 10\omega^2 + 9} = \frac{\omega^2}{(\omega^2+1)(\omega^2+9)}.$$

$\omega^2 = x$

By partial fractions, we have

$$\frac{\omega^2}{(\omega^2+1)(\omega^2+9)} = \frac{A}{\omega^2+1} + \frac{B}{\omega^2+9}$$

$$\omega^2 = A(\omega^2+9) + B(\omega^2+1)$$

$$\omega^2 = -1 \Rightarrow -1 = A(\cancel{-1}\!\!-1+9) + B \times 0$$

$$-1 = A(8)$$

$$A = \frac{-1}{8}$$

$$\omega^2 = -9 \Rightarrow -9 = A(-9+9) + B(-9+1)$$

$$B = 9/8$$

$$\frac{\omega^2}{(\omega^2+1)(\omega^2+9)} = \frac{-1/8}{(\omega^2+1)} + \frac{9/8}{(\omega^2+9)}$$

$$F^{-1}[S_{XX}(\omega)] = F^{-1}\left(\frac{-1/8}{\omega^2+1} + \frac{9/8}{\omega^2+9}\right)$$

By linearity property, we have

$$= \frac{-1}{8} F^{-1}\left[\frac{1}{1+\omega^2}\right] + \frac{9}{8} F^{-1}\left[\frac{1}{3^2+\omega^2}\right]$$

We know $\quad e^{-b|\tau|} \xleftrightarrow{F.T} \dfrac{2b}{b^2+\omega^2}$

$b=1 \Rightarrow \quad e^{-|\tau|} \longleftrightarrow \dfrac{2}{1+\omega^2}$

$$\frac{1}{2}e^{-|\tau|} \longleftrightarrow \frac{1}{1+\omega^2}$$

$$b=3 \Rightarrow \quad e^{-3|\tau|} \longleftrightarrow \frac{2\times 3}{3^2+\omega^2}$$

$$\frac{1}{6}\; e^{-3|\tau|} \longleftrightarrow \frac{1}{3^2+\omega^2}$$

$$R_{xx}(\tau) \longleftrightarrow S_{xx}(\omega)$$

$$R_{xx}(\tau) = F^{-1}[S_{xx}(\omega)] = \frac{-1}{8}\cdot\frac{1}{2}\,e^{-|\tau|} + \frac{\cancel{9}^{3}}{8}\cdot\frac{1}{\cancel{6}_{2}}\,e^{-3|\tau|}$$

$$= -\frac{1}{16}e^{-|\tau|} + \frac{3}{16}\,e^{-3|\tau|}$$

$$\boxed{\therefore R_{xx}(\tau) = \frac{1}{8}\left[\frac{3}{2}\,e^{-3|\tau|} - \frac{1}{2}\,e^{-|\tau|}\right]}$$

7. $$S_{xx}(\omega) = \frac{\omega^2}{\omega^4+13\omega^2+36} =$$

$$S_{xx}(\omega) = \frac{\omega^2}{\omega^4+13\omega^2+36} = \frac{\omega^2}{(\omega^2+4)(\omega^2+9)}$$

By partial fractions, we have

$$\frac{\omega^2}{(\omega^2+4)(\omega^2+9)} = \frac{A}{\omega^2+4} + \frac{B}{(\omega^2+9)}$$

$$\omega^2 = A(\omega^2+9) + B(\omega^2+4)$$

$$\omega^2 = -4 \Rightarrow -4 = A(-4+9) + B(-4+4)$$

$$-4 = A(5)$$

$$A = -\frac{4}{5}$$

$$A = -\frac{4}{5}$$

$$\omega^2 = -9 \Rightarrow \quad -9 = A(-9+9) + B(-9+4)$$

$$B = \frac{9}{5}$$

$$\frac{\omega^2}{\omega^4+13\omega^2+36} = \frac{-\frac{4}{5}}{\omega^2+4} + \frac{\frac{9}{5}}{\omega^2+9}$$

$$F^{-1}[S_{xx}(\omega)] = F^{-1}\left[\frac{-4/5}{\omega^2+4} + \frac{9/5}{\omega^2+9}\right]$$

By linearity property, we have

$$= -\frac{4}{5} F^{-1}\left[\frac{1}{\omega^2+4}\right] + \frac{9}{5} F^{-1}\left(\frac{1}{\omega^2+9}\right)$$

We know $\quad e^{-b|\tau|} \xleftrightarrow{F.T.} \dfrac{2b}{b^2+\omega^2}$

$$b = 2 \Rightarrow \quad e^{-2|\tau|} \longleftrightarrow \frac{2(2)}{4+\omega^2}$$

$$\frac{1}{4} e^{-2|\tau|} \longleftrightarrow \frac{1}{\omega^2+4}$$

$$b = 3 \Rightarrow \quad e^{-3|\tau|} \longleftrightarrow \frac{2 \times 3}{9+\omega^2}$$

$$\frac{1}{6} e^{-3|\tau|} \longleftrightarrow \frac{1}{\omega^2+9}$$

$$R_{xx}(\tau) = S_{xx}(\omega)$$

$$R_{xx}(\tau) = F^{-1}[S_{xx}(\omega)] = -\frac{4}{5} \cdot \frac{1}{4} e^{-2|\tau|} + \frac{9}{5} \cdot \frac{1}{6} e^{-3|\tau|}$$

$$R_{xx}(\tau) = -\frac{1}{5} e^{-2|\tau|} + \frac{3}{10} e^{-3|\tau|}$$

$$\boxed{R_{xx}(\tau) = \frac{1}{5}\left[\frac{3}{2} e^{-3|\tau|} - e^{-2|\tau|}\right]}$$

8. Find ACF, P_{xx} and RMS bandwidth of rps of $x(t)$

$$S_{xx}(\omega) = \begin{cases} \pi & ; \ |\omega| \le 1 \\ 0 & ; \ \text{otherwise} \end{cases}$$

Sol: Given $S_{xx}(\omega) = \begin{cases} \pi & ; \ -1 \le \omega \le 1 \\ 0 & ; \ \text{otherwise} \end{cases}$

(i) The ACF of $X(t) = R_{xx}(\tau) = F^{-1}[S_{xx}(\omega)]$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega) \cdot e^{j\omega\tau}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-1}^{1} \pi\, e^{j\omega\tau}\, d\omega$$

$$= \frac{\pi}{2\pi}\int_{-1}^{1} e^{j\omega\tau}\, d\omega$$

$$= \frac{1}{2}\left[\frac{e^{j\omega\tau}}{j\tau}\right]_{-1}^{1}$$

$$= \frac{1}{2j\tau}\left[e^{j\tau} - e^{-j\tau}\right]$$

$$= \frac{1}{\tau}\left(\frac{e^{j\tau} - e^{-j\tau}}{2j}\right)$$

$$= \frac{\tau}{\tau} \cdot \frac{\sin(\tau)}{\tau}$$

$$R_{xx}(\tau) = Sa(\tau)$$

(ii) Average power $= P_{xx} = R_{xx}(0) = Sa(1) = 1$

(iii) RMS Band width

$$\omega_{rms}^2 = \frac{\int_{-\infty}^{\infty} \omega^2 \cdot S_{xx}(\omega)\, d\omega}{\int_{-\infty}^{\infty} S_{xx}(\omega)\, d\omega}$$

$$\int_{-\infty}^{\infty} S_{XX}(\omega)\, d\omega = 2\pi P_{XX} = 2\pi \cdot 1 = 2\pi$$

$$\int_{-\infty}^{\infty} \omega^2 \cdot S_{XX}(\omega)\, d\omega = \int_{-1}^{1} \omega^2 \cdot \pi \cdot d\omega$$

$$= \pi \left[\frac{\omega^3}{3}\right]_{-1}^{1}$$

$$= \pi \left[\frac{1^3 - (-1)^3}{3}\right]$$

$$= \pi \left[\frac{2}{3}\right]$$

$$= \frac{2\pi}{3}$$

$$\omega^2_{rms} = \frac{2\pi}{3} \times \frac{1}{2\pi}$$

$$\omega^2_{rms} = \frac{1}{3}$$

$\therefore$ RMS bandwidth $= \omega_{rms} = \frac{1}{\sqrt{3}}$ rad/sec

9. Find ACF, P_{XX} and RMS bandwidth of PSD of $X(t)$

$$S_{XX}(\omega) = \begin{cases} 1 - \frac{\omega}{4\pi} & ; |\omega| \leq 4\pi \\ 0 & ; \text{otherwise} \end{cases}$$

Sol: Given $S_{XX}(\omega) = \begin{cases} 1 - \frac{\omega}{4\pi} & ; -4\pi \leq \omega \leq 4\pi \\ 0 & ; \text{otherwise} \end{cases}$

(i) ACF of $X(t) = R_{XX}(\tau) = F^{-1}[S_{XX}(\omega)]$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega)\, e^{j\omega\tau}\, d\omega$$

$$= \frac{1}{2\pi} \int_{-4\pi}^{4\pi} \left(1 - \frac{\omega}{4\pi}\right) e^{j\omega\tau}\, d\omega$$

$$= \frac{1}{2\pi}\left[\left(1-\frac{\omega}{4\pi}\right)\left(\frac{e^{j\omega\tau}}{j\tau}\right)-\left(\frac{-1}{4\pi}\right)\cdot\frac{e^{j\omega\tau}}{j\tau\times j\tau}\right]_{-4\pi}^{4\pi}$$

$$= \frac{1}{2\pi}\left\{\left[\left(1-\frac{4\pi}{4\pi}\right)\frac{e^{j4\pi\tau}}{j\tau}\right]+\frac{1}{4\pi}\frac{e^{j4\pi\tau}}{(j\tau)^2} - \left[2\left(\frac{e^{-4\pi\tau}}{j\tau}\right)+\frac{1}{4\pi}\frac{e^{j(-4\pi)\tau}}{(j\tau)^2}\right]\right]$$

$$= \frac{1}{2\pi}\left(0+\frac{1}{4\pi}\frac{e^{j4\pi\tau}}{(j\tau)^2}-2\frac{e^{-4\pi j\tau}}{j\tau} - \frac{1}{4\pi}\frac{e^{j(-4\pi)\tau}}{(j\tau)^2}\right]$$

$$= \frac{1}{2\pi}\left(\frac{1}{4\pi(j\tau)^2}\left[e^{4\pi j\tau}-e^{-4\pi j\tau}\right]-\frac{2}{j\tau}e^{-4\pi j\tau}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{2\pi j\tau^2}\left(\frac{e^{4\pi j\tau}-e^{4\pi j\tau}}{2j}\right)-\frac{2}{j\tau}e^{-4\pi j\tau}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{2\pi j\tau^2}\left(\frac{\sin 4\pi\tau}{4\pi\tau}\right)\right.$$

Average power $P_{xx} = E[x^2(t)]$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)\,d\omega$$

$$= \frac{1}{2\pi}\int_{-4\pi}^{4\pi}\left(1-\frac{\omega}{4\pi}\right)d\omega$$

$$= \frac{1}{2\pi}\left[\omega-\frac{\omega^2}{8\pi}\right]_{-4\pi}^{4\pi}$$

$$= \frac{1}{2\pi}\left[8\pi-\frac{1}{8\pi}(32\pi^2)\right]$$

$$= \frac{1}{2\pi}[8\pi]$$

$\therefore P_{xx} = 4$ watts

(iii) RMS bandwidth:

$$\omega^2_{rms} = \frac{\int_{-\infty}^{\infty} \omega^2 S_{XX}(\omega)\,d\omega}{\int_{-\infty}^{\infty} S_{XX}(\omega)\,d\omega}$$

$$\int_{-\infty}^{\infty} S_{XX}(\omega)\,d\omega = 2\pi P_{XX} = 8\pi$$

$$\int_{-\infty}^{\infty} \omega^2 S_{XX}(\omega)\,d\omega = \int_{-4\pi}^{4\pi} \omega^2\left(1 - \frac{\omega}{4\pi}\right)d\omega$$

$$= \int_{-4\pi}^{4\pi}\left(\omega^2 - \frac{\omega^3}{4\pi}\right)d\omega$$

$$= \left[\frac{\omega^3}{3}\right]_{4\pi}^{4\pi} - \frac{1}{4\pi}\left[\frac{\omega^4}{4}\right]_{4\pi}^{4\pi}$$

$$= \frac{(4\pi)^3 - (4\pi)^3}{3} - \frac{1}{4\pi}\left(\frac{4\pi^4 - 4\pi^4}{4}\right)$$

$$= \frac{128\pi^3}{3}$$

$$\therefore \int_{-\infty}^{\infty} \omega^2 S_{XX}(\omega)\,d\omega = \frac{128\pi^3}{3}$$

$$\omega^2_{rms} = \frac{\frac{128\pi^3}{3}}{8\pi}$$

$$= \frac{128\pi^3}{3} \times \frac{1}{8\pi}$$

$$\omega^2_{rms} = \frac{16\pi^2}{3}$$

$$\therefore \omega_{rms} = \frac{4\pi}{\sqrt{3}} \text{ rad/sec}$$

*Cross Power Density Spectrum:

Definition ①: let $X_T(\omega)$ and $Y_T(\omega)$ are fourier transform of random processes $X(t)$ and $Y(t)$ in the interval $[-T, T]$. Then the cross power density spectrum is defined by

The Cross PSD of $X(t)$ & $X(t)$ $= S_{xy}(\omega) = \underset{T\to\infty}{Lt} \dfrac{E[X_T^*(\omega)\ Y_T(\omega)]}{2T}$

$$|||^{ly} \quad S_{yx}(\omega) = \underset{T\to\infty}{Lt} \frac{E[Y_T^*(\omega)\ X_T(\omega)]}{2T}$$

$$\because S_{xx}(\omega) = \underset{T\to\infty}{Lt} \frac{E(|X_T(\omega)|^2)}{2T} = \underset{T\to\infty}{Lt} \frac{E[X_T^*(\omega) X_T(\omega)]}{2T}$$

Definition ②: let X_T and Y_T are two real random processes and are jointly WSS processes. Then the cross power density spectrum between two processes is defined as the fourier transform of cross-correlation between two processes, i.e.,

The cross PSD of $X(t)$ & $Y(t)$ $= S_{xy}(\omega) \xleftrightarrow{F.T}$

$R_{xy}(\tau)$ = The CCF b/n $X(t)$ & $Y(t)$

$$S_{xy}(\omega) = F[R_{xy}(\tau)] = \int_{-\infty}^{\infty} R_{xy}(\tau)\ e^{-j\omega\tau} d\tau$$

Similarly, $S_{yx}(\omega) = F[R_{yx}(\tau)] = \int_{-\infty}^{\infty} [R_{yx}(\tau)] e^{-j\omega\tau} d\tau$

*Properties of Cross Power Density Spectrum:

1. Cross power density spectrum satisfies complex conjugate symmetry i.e, $S_{xy}(\omega) = S_{yx}(-\omega) = S_{yx}^*(\omega)$

Proof: We know, $S_{xy}(\omega) = F[R_{xy}(\tau)] = \int_{-\infty}^{\infty} [R_{xy}(\tau)]\ e^{-j\omega\tau} d\tau$

$$S_{yx}(\omega) = F[R_{yx}(\tau)] = \int_{-\infty}^{\infty} [R_{yx}(\tau)]\ e^{-j\omega\tau} d\tau$$

Replace ω by $-\omega$, then we have

$$S_{YX}(-\omega) = \int_{-\infty}^{\infty} R_{YX}(\tau)\, e^{-j(-\omega)\tau} d\tau$$

$$= \int_{-\infty}^{\infty} R_{YX}(\tau)\, e^{j\omega\tau}\, d\tau$$

Put $\tau = -t \Rightarrow d\tau = -dt$

$$= \int_{\infty}^{-\infty} R_{YX}(-t)\, e^{j\omega(-t)} (-dt)$$

$$= (-1)\int_{-\infty}^{\infty} R_{YX}(-t)\, e^{-j\omega t} (-dt)$$

[Interchanging the limits]

$$= \int_{-\infty}^{\infty} R_{YX}(-\tau)\, e^{-j\omega\tau}\, d\tau \quad \because t = \tau$$

We know $R_{XY}(\tau) = R_{YX}(-\tau)$

$$= \int_{-\infty}^{\infty} R_{XY}(\tau) \cdot e^{-j\omega\tau} d\tau = F[R_{XY}(\tau)]$$

$$\boxed{S_{YX}(-\omega) = S_{XY}(\omega)}$$

Hence (i) proved.

2. $$S_{YX}(\omega) = F[R_{YX}(\tau)] = \int_{-\infty}^{\infty} [R_{YX}(\tau)] e^{-j\omega\tau} d\tau$$

Apply complex conjugate on both sides

$$[S_{YX}(\omega)]^* = \left[\int_{-\infty}^{\infty} R_{YX}(\tau)\, e^{-j\omega\tau}\, d\tau\right]^*$$

$$S_{YX}^*(\omega) = \int_{-\infty}^{\infty} \left(R_{YX}(\tau)\, e^{-j\omega\tau}\right)^* d\tau$$

$$= \int_{-\infty}^{\infty} R_{YX}^*(\tau)\, (e^{-j\omega\tau})^*\, d\tau$$

$$S_{YX}^{*}(\omega) = \int_{-\infty}^{\infty} R_{YX}^{*}(\tau) \cdot e^{j\omega\tau} d\tau \qquad \because (e^{-j\theta})^{*} = e^{j\theta}$$

We know $R_{YX}(\tau) = R_{YX}^{*}(\tau)$

$\boxed{\because R_{YX}(-\tau) = R_{XY}(\tau)}$

$$S_{YX}^{*}(\omega) = \int_{-\infty}^{\infty} R_{YX}(\tau) \, e^{j\omega\tau} d\tau$$

Put $\tau = -t$

$$= \int_{\infty}^{-\infty} R_{YX}(-t) \, e^{j\omega(-t)}(-dt)$$

$$= -\int_{-\infty}^{\infty} R_{YX}(-\tau) \, e^{-j\omega\tau}(-d\tau) \qquad \because t = \tau$$

$$= \int_{-\infty}^{\infty} R_{YX}(-\tau) \, e^{-j\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} R_{XY}(\tau) \, e^{-j\omega\tau} d\tau$$

$$\therefore \boxed{S_{YX}(\omega) = S_{XY}(\omega)}$$

Hence proved.

2. Real component of $S_{XY}(\omega)$ ~~satisfies~~ is an even function of ω and real component of $S_{YX}(\omega)$ is also an even function of ω.

Proof: We know $S_{XY}(\omega) = F[R_{XY}(\tau)] = \int_{-\infty}^{\infty} [R_{XY}(\tau)] e^{-j\omega\tau} d\tau$

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau) \, [\cos\omega\tau - j\sin\omega\tau] \, d\tau$$

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau) \cos(\omega\tau) \, d\tau - j\int_{-\infty}^{\infty} R_{XY}(\tau) \sin(\omega\tau) \, d\tau$$

$\{\because e^{-j\theta} = \cos\theta - j\sin\theta\}$

Real component of $S_{xy}(\omega) = \int_{-\infty} R_{xy}(\tau) \cos\omega\tau \, d\tau$

Imaginary component of $S_{xy}(\omega) = -\int_{-\infty}^{\infty} R_{xy}(\tau) \sin\omega\tau \, d\tau$

$Re[S_{xy}(\omega)]$ containing the term $\cos(\omega\tau)$ i.e., even function of ω

$\therefore$ $Re[S_{xy}(\omega)]$ is an even function of 'ω'.

Similarly $S_{yx}(\omega) = \int_{-\infty}^{\infty} [R_{yx}(\tau)] e^{-j\omega\tau} \, d\tau$

$$= \int_{-\infty}^{\infty} R_{yx}(\tau) \left[\cos\omega\tau - j\sin\omega\tau\right] d\tau$$

$$S_{yx}(\omega) = \int_{-\infty}^{\infty} R_{yx}(\tau) \cos\omega\tau \, d\tau - j\int_{-\infty} R_{yx}(\tau) \sin\omega\tau \, d\tau$$

Real component of $S_{yx}(\omega) = \int_{-\infty}^{\infty} R_{yx}(\tau) \cos\omega\tau \, d\tau$

Imaginary component of $S_{yx}(\omega) = -\int_{-\infty}^{\infty} R_{yx}(\tau) \sin\omega\tau \, d\tau$

$Re[S_{yx}(\omega)]$ containing the term $\cos\omega\tau$ i.e. evenfunction of ω.

$$Re[S_{xy}(-\omega)] = \int_{-\infty}^{\infty} R_{xy}(\tau) \cos[(-\omega)\tau] \, d\tau$$

$$= \int_{-\infty}^{\infty} R_{xy}(\tau) \cos(\omega\tau) \, d\tau$$

$$= Re[S_{xy}(\omega)$$

$\boxed{\therefore Re[S_{xy}(-\omega)] = Re[S_{xy}(\omega)]}$ which is a even function of

$|||^{ly}$ $Re[S_{yx}(\omega)] = \int_{-\infty}^{\infty} R_{yx}(\tau) \cos(\omega\tau) \, d\tau$

$$Re[S_{xy}(-\omega)] = \int_{-\infty}^{\infty} R_{yx}(\tau) \cos((\omega)\tau) \, d\tau$$

$$= \int_{-\infty}^{\infty} R_{yx}(\tau) \cos\omega\tau \, d\tau$$

$$\therefore \boxed{Re[S_{yx}(-\omega)] = Re[S_{yx}(\omega)]}$$

$$S_{yx}(\omega) = F[R_{yx}(\tau)] = \int_{-\infty}^{\infty} R_{yx}(\tau) e^{-j\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} R_{yx}(\tau) \cos\omega\tau \, d\tau - j\int_{-\infty}^{\infty} R_{yx}(\tau) \sin\omega\tau \, d\tau$$

$$Re[S_{yx}(\omega)] = \int_{-\infty}^{\infty} R_{yx}(\tau)\cos\omega\tau \, d\tau$$

$$Im[S_{yx}(\omega)] = -\int_{-\infty}^{\infty} R_{yx}(\tau) \sin\omega\tau \, d\tau$$

3. Imaginary part of $S_{xy}(\omega)$ and $S_{yx}(\omega)$ are odd function of ω. i.e $Im[S_{xy}(\omega)] = -Im[S_{xy}(-\omega)]$ and

$$Im[S_{yx}(\omega)] = -Im[S_{yx}(-\omega)]$$

Proof: let, $Im[S_{xy}(\omega)] = -\int_{-\infty}^{\infty} R_{xy}(\tau) \sin(\omega\tau) \, d\tau$

$$Im[S_{xy}(-\omega)] = -\int_{-\infty}^{\infty} R_{xy}(\tau) \sin[(-\omega)\tau] d\tau$$

$$= -\int_{-\infty}^{\infty} R_{xy}(\tau) \; -\sin\omega\tau \, d\tau$$

$$= \int_{-\infty}^{\infty} R_{xy}(\tau) \sin\omega\tau \, d\tau$$

$$= -Im[S_{xy}(\omega)]$$

$$\therefore \boxed{Im[S_{xy}(\omega)] = -Im[S_{xy}(\omega)]}$$

IIIly $\operatorname{Im}[S_{YX}(\omega)] = -\int_{-\infty}^{\infty} R_{YX}(\tau) \sin(\omega\tau)\, d\tau$

$$\operatorname{Im}[S_{YX}(-\omega)] = -\int_{-\infty}^{\infty} R_{YX}(\tau) \sin(-\omega\tau)\, d\tau$$

$$= -\int_{-\infty}^{\infty} R_{YX}(\tau) - \sin(\omega\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} R_{YX}(\tau) \sin\omega\tau\, d\tau$$

$$\boxed{\therefore \operatorname{Im}[S_{YX}(-\omega)] = -\operatorname{Im}[S_{YX}(\omega)]}$$

4. If $X(t)$ and $Y(t)$ are orthogonal random processes then $S_{XY}(\omega) = 0$ and $S_{YX}(\omega) = 0$

Proof: The CrossPSD of $X(t)$ & $Y(t) = S_{XY}(\omega) = F[R_{XY}(\tau)]$

$$= \int_{-\infty}^{\infty} R_{XY}(\tau)\, e^{-j\omega\tau}\, d\tau$$

Also We know $S_{YX}(\omega) = F[R_{YX}(\tau)] = \int_{-\infty}^{\infty} R_{YX}(\tau)\, e^{-j\omega\tau} d\tau$

We know that $X(t)$ and $Y(t)$ are said to be two orthogonal random processes if and only if their cross-correla function is zero, i.e., $R_{XY}(\tau) = R_{YX}(\tau) = 0$

$$\therefore S_{XY}(\omega) = \int_{-\infty}^{\infty} (0) \cdot e^{-j\omega\tau} d\tau \Rightarrow S_{XY}(\omega) = 0$$

$$S_{YX}(\omega) = \int_{-\infty}^{\infty} (0)\, e^{-j\omega\tau} d\tau \Rightarrow S_{YX}(\omega) = 0$$

5. If two random processes are uncorrelated and have constant mean values $\overline{X(t)}$ and $\overline{Y(t)}$ then

$$S_{XY}(\omega) = S_{YX}(\omega) = 2\pi \bar{X}\bar{Y}\delta(\omega)$$

Proof:

The cross PSD of $X(t)$ and $Y(t)$ = $S_{XY}(\omega) = F[R_{XY}(\tau)] = \int_{-\infty}^{\infty} R_{XY}(\tau) e^{-j\omega\tau} d\tau$

$$S_{YX}(\omega) = F[R_{YX}(\tau)] = \int_{-\infty}^{\infty} R_{YX}(\tau) e^{-j\omega\tau} d\tau$$

We know $R_{XY}(\tau) = E[X(t) \cdot Y(t+\tau)]$

If $X(t)$ and $Y(t)$ are uncorrelated or independent random processes then $E[X(t)\, Y(t+\tau)] = E[X(t)]\, E[Y(t+\tau)]$

$$S_{XY}(\tau) = \int_{-\infty}^{\infty} E[X(t)\; Y(t+\tau)]\, e^{-j\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} E[X(t)]\; E[Y(t+\tau)]\; e^{-j\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} \bar{X} \cdot \bar{Y}\; e^{-j\omega\tau} d\tau \qquad \because E[X(t)] = \bar{X};\ E[Y(t+\tau)] = \bar{Y}.$$

$$= \bar{X} \cdot \bar{Y} \int_{-\infty}^{\infty} 1 \cdot e^{-j\omega\tau} d\tau \qquad \because \int_{-\infty}^{\infty} 1 \cdot e^{-j\omega\tau} d\tau = 2\pi\,\delta(\omega)$$

$$= \bar{X} \cdot \bar{Y} \cdot 2\pi\,\delta(\omega)$$

$$\therefore S_{XY}(\tau) = 2\pi\,\bar{X}\,\bar{Y}\,\delta(\omega)$$

Hence proved.

6. Derive relationship b/n cross power spectrum density and cross-correlation function.

Proof:

Statement: The cross-correlation function and the cross power spectrum density form a fourier transform pair.

The CCF b/n $X(t)$ & $Y(t) = Rxy(\tau) \xleftrightarrow{F.T} Sxy(\omega)$ = The PSD of $X(t)$ & $Y(t)$

i.e, $Sxy(\omega) = F[A[Rxy(t, t+\tau)]]$

For WSS random process, $Sxy(\omega) = F[Rxy(\tau)]$

Proof: ~~Re~~

Let $X(t)$ and $Y(t)$ are two real random processes $X_T(t)$ and $Y_T(t)$ be defined in the interval $[-T, T]$ then the fourier transform of functions be defined as

$$X_T(\omega) = F[X_T(t)] = \int_{-\infty}^{\infty} X_T(t)\, e^{-j\omega t}\, dt = \int_{-T}^{T} X(t)\, e^{-j\omega t} dt$$

$$Y_T(\omega) = F[Y_T(t)] = \int_{-\infty}^{\infty} Y_T(t_1)\, e^{-j\omega t_1} dt_1 = \int_{-T}^{T} Y(t_1)\, e^{-j\omega t_1} dt_1$$

$\{\because X_T(t) = X(t)\ ;\ -T \le t \le T$
$Y_T(t_1) = Y(t_1)\ ;\ -T \le t_1 \le T\}$

$$X_T^*(\omega) = \left[\int_{-T}^{T} X(t) \cdot e^{-j\omega t} dt\right]^*$$

$$= \int_{-T}^{T} X^*(t) \cdot (e^{-j\omega t})^* dt$$

$$= \int_{-T}^{T} X(t) \cdot e^{j\omega t} dt \qquad \{\because X^*(t) = X(t)\ ;\ (e^{-j\theta})^* = e^{j\theta}$$

$$= \int_{-T}^{T} X(t) \cdot e^{j\omega t} dt$$

The cross PSD is defined by $Sxy(\omega) = \lim_{T\to\infty} \frac{E[X_T^*(\omega)\, Y_T(\omega)]}{2T}$

$$Sxy(\omega) = \lim_{T\to\infty} \frac{E\left[\int_{-T}^{T} X(t)\, e^{j\omega t} dt \cdot Y_T(\omega)\right]}{2T}$$

$$= \lim_{T\to\infty} \frac{E\left[\int_{-T}^{T} X(t)\, e^{j\omega t} dt \int_{-T}^{T} Y(t_1)\, e^{-j\omega t_1} dt_1\right]}{2T}$$

$$= \underset{T\to\infty}{Lt} \frac{1}{2T} E \int_{-T}^{T}\int_{-T}^{T} X(t)\, Y(t_1)\, e^{-j\omega(t_1-t)}\, dt_1\, dt$$

$$= \underset{T\to\infty}{Lt} \frac{1}{2T} \int_{-T}^{T}\int_{-T}^{T} E\left[X(t)\, Y(t_1)\right] e^{-j\omega(t_1-t)}\, dt_1\, dt$$

$$S_{XY}(\omega) = \underset{T\to\infty}{Lt} \frac{1}{2T} \int_{-T}^{T}\int_{-T}^{T} R_{XY}(t, t_1)\, e^{-j\omega(t_1-t)}\, dt_1\, dt$$

$\because R_{XY}(t, t_1) = E[X(t)Y(t_1)]$

Apply $\frac{1}{2\pi}\int_{-\infty}^{\infty} (\quad) e^{j\omega\tau} d\omega$, we get

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega)\, e^{j\omega\tau} d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{XY}(t, t_1)\, e^{-j\omega(t_1-t)}\, dt_1\, dt\right] e^{j\omega\tau} d\omega$$

$$= \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{XY}(t, t_1)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} (1)\, e^{-j\omega(t_1-t-\tau)} d\omega\right] dt_1\, dt$$

$\because \frac{1}{2\pi}\int_{-\infty}^{\infty} (1)\, e^{-j\omega\tau} d\omega = \delta(\tau)$

$\frac{1}{2\pi}\int_{-\infty}^{\infty} (1)\, e^{-j\omega(t_1-t-\tau)} d\omega = \delta(t_1 - t - \tau)$

$$= \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T}\left[\int_{-T}^{T} R_{XY}(t, t_1)\, \delta(t_1 - t - \tau)\right] dt$$

$\because \int_{-T}^{T} f(t_1)\, \delta(t_1 - T_0)\, dt_1 = f(t_1)\Big|_{t_1 = T_0} = f(T_0)$

$$= \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T}\left(R_{XY}(t, t_1)\Big|_{t_1 = t+\tau}\right) dt$$

$$= \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T} R_{XY}(t, t+\tau)\, dt$$

$$\Rightarrow A\left[R_{XY}(t, t+\tau)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[S_{XY}(\omega)\right] e^{j\omega\tau} d\omega$$

$$F^{-1}[S_{xy}(\omega)] = A[R_{xy}(t, t+\tau)]$$

Apply fourier transform on both sides, we get

$$F[F^{-1}[S_{xy}(\omega)]] = F[A[R_{xy}(t, t+\tau)]]$$

$$\boxed{\therefore S_{xy}(\omega) = F[A[R_{xy}(t, t+\tau)]]}$$

If $x(t)$ and $y(t)$ are jointly WSS, then

$$A[R_{xy}(t, t+\tau)] = R_{xy}(\tau)$$

$$S_{xy}(\omega) = F[A[R_{xy}(t, t+\tau)]]$$

$$S_{xy}(\omega) = F[R_{xy}(\tau)] = \int_{-\infty}^{\infty} R_{xy}(\tau) e^{-j\omega\tau} d\tau$$

Hence proved.

* Problems:

1. Let the cross PSD is $S_{xy}(\omega) = \frac{1}{25+\omega^2}$. Find CCF.

Sol: Given $S_{xy}(\omega) = \frac{1}{25+\omega^2}$

We know relation b/n CPSD and CCF as

$$S_{xy}(\omega) = F[R_{xy}(\tau)] = \int_{-\infty}^{\infty} R_{xy}(\tau) e^{-j\omega\tau} d\tau$$

Symbolically $R_{xy}(\tau) \xleftrightarrow{F.T} S_{xy}(\omega)$

$$\Rightarrow e^{-b|\tau|} \longleftrightarrow \frac{2b}{b^2+\omega^2}$$

$$b = 5; \quad e^{-5|\tau|} \longleftrightarrow \frac{2\times 5}{5^2+\omega^2}$$

$$\frac{1}{10} e^{-5|\tau|} \longleftrightarrow \frac{1}{25+\omega^2}$$

CCF b/n $x(t)$ and $y(t) = R_{xy}(\tau) = F^{-1}[S_{xy}(\omega)]$

$$= F^{-1}\left[\frac{1}{25+\omega^2}\right].$$

$$\boxed{\therefore R_{xy}(\tau) = \frac{1}{10} e^{-5|\tau|}}$$

2. Given $S_{xy}(\omega) = \frac{1}{(a+j\omega)^2}$. Find CCF.

Sol: Given $S_{xy}(\omega) = \frac{1}{(a+j\omega)^2}$

$$S_{xy}(\omega) = F[R_{xy}(\tau)] = \int_{-\infty}^{\infty} R_{xy}(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$R_{xy}(\tau) \xleftrightarrow{FT} S_{xy}(\omega)$$

We know $e^{-at} u(t) \longleftrightarrow \frac{1}{a+j\omega}$

$$R_{xy}(t) = t\, e^{-at} u(t) \longleftrightarrow \frac{1}{(a+j\omega)^2} = S_{xy}(\omega)$$

CCF b/n $x(t)$ and $y(t) = R_{xy}(\tau) = F^{-1}[S_{xy}(\omega)]$

$$= F^{-1}\left[\frac{1}{(a+j\omega)^2}\right].$$

$$\boxed{\therefore R_{xy}(\tau) = \tau\, e^{-a\tau} u(\tau)}$$

3. Given $S_{xy}(\omega) = \begin{cases} k + \frac{j\omega}{W} & ; |\omega| \le W \\ 0 & ; \text{elsewhere} \end{cases}$. Find CCF.

Sol: Given $S_{xy}(\omega) = \begin{cases} k + \frac{j\omega}{W} & ; -W \le \omega \le W \\ 0 & ; \text{elsewhere} \end{cases}$

$$K_{xy}(\tau) = F\left[S_{xy}(\omega)\right] = \frac{1}{2\pi}\int_{-\infty}\left[S_{xy}(\omega)\right] e^{j\omega\tau}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(K + \frac{j\omega}{W}\right) e^{j\omega\tau}\, d\omega$$

$$= \frac{K}{2\pi}\int_{-W}^{W} 1\cdot e^{j\omega\tau}\, d\omega + \frac{j}{2\pi W}\int_{-W}^{W} \omega\cdot e^{j\omega\tau}\, d\omega$$

$$= \frac{K}{2\pi j\tau}\left[e^{j\omega\tau} - e^{-j\omega\tau}\right] + \frac{j}{2\pi W}\left[W\left(\frac{e^{j\omega\tau}}{j\tau}\right) - \frac{e^{j\omega\tau}}{(j\tau)^2} - \left(\frac{W e^{-j\omega\tau}}{j\tau} - \frac{e^{-j\omega\tau}}{(j\tau)^2}\right)\right]$$

$$= \frac{K}{\pi\tau}\left(\frac{e^{j\omega\tau} - e^{-j\omega\tau}}{2j}\right) + \frac{j}{2\pi W}\left[\frac{W}{j\tau}\left(e^{j\omega\tau} + e^{-j\omega\tau}\right) - \frac{1}{(j\tau)^2}\left[e^{j\omega\tau} - e^{-j\omega\tau}\right]\right]$$

$$= \frac{K}{\pi\tau}(\sin\omega\tau) + \frac{j}{2\pi W j\tau}\left[W\left(e^{j\omega\tau} + e^{-j\omega\tau}\right)\right] - \frac{j}{2\pi W}\left(\frac{e^{j\omega\tau} - e^{-j\omega\tau}}{(j\tau)^2}\right)$$

$$= \frac{K}{\pi\tau}(\sin\omega\tau) + \frac{j}{2\pi\omega j\tau}\, W\cdot 2\cos(\omega\tau) - \frac{j}{j^2 2\pi W\tau^2}\, 2j\sin\omega\tau$$

$$= \frac{K}{\pi\tau}(\sin\omega\tau) + \frac{\cancel{j}}{\cancel{2}\pi\cancel{W}\cancel{j}\tau}\cdot\cancel{W}\cdot 2\cos(\omega\tau) - \frac{\cancel{j}}{j^{\cancel{2}}2\pi W\tau^2}\cdot\cancel{2}\cancel{j}\sin\omega\tau$$

$$= \frac{K}{\pi\tau} \sin(\omega\tau) - \frac{1}{\pi W \tau^2}\left(\frac{e^{j\omega\tau} - e^{-j\omega\tau}}{2j}\right) + \frac{1}{\pi\tau}\cos\omega\tau$$

$$R_{xy}(\tau) = \left[\frac{K}{\pi\tau} - \frac{1}{\pi W \tau^2}\right] \sin(\omega\tau) + \frac{1}{\pi\tau}\cos\omega\tau$$

4. Given $R_{xy}(\tau) = K e^{-k|\tau|}$, then show that

$$S_{xy}(\omega) = \frac{2}{1+\left(\frac{\omega}{k}\right)^2}$$

Sol: Given $R_{xy}(\tau) = k e^{-k|\tau|}$

$$S_{xy}(\omega) = \frac{2}{1+\left(\frac{\omega}{k}\right)^2}$$

$$S_{xy}(\omega) = F[R_{xy}(\tau)] = \int_{-\infty}^{\infty} R_{xy}(\tau) e^{-j\omega\tau} d\tau \rightarrow ①$$

$$R_{xy}(\tau) \xleftrightarrow{FT} S_{xy}(\omega)$$

$$e^{-b|\tau|} \longleftrightarrow \frac{2b}{b^2+\omega^2}$$

$$b=k; \quad e^{-k|\tau|} \longleftrightarrow \frac{2k}{k^2+\omega^2}$$

$$① \Rightarrow S_{xy}(\omega) = \int_{-\infty}^{\infty} k e^{-k|\tau|} e^{-j\omega\tau} d\tau$$

$$= k\left[\int_{-\infty}^{0} e^{k\tau} e^{-j\omega\tau} d\tau + \int_{0}^{\infty} e^{-k\tau} e^{-j\omega\tau} d\tau\right]$$

$$= k\left[\int_{-\infty}^{0} e^{(k-j\omega)\tau} d\tau + \int_{0}^{\infty} e^{-(k-j\omega)\tau} d\tau\right]$$

$$= k\left[\left[\frac{e^{(k-j\omega)\tau}}{(k-j\omega)}\right]_{-\infty}^{0} + \left[\frac{e^{-(k-j\omega)\tau}}{-(k+j\omega)}\right]_{0}^{\infty}\right]$$

$$= k\left(\frac{1}{k-j\omega}\left[e^{(k-j\omega)(0)} - e^{(k-j\omega)(-\infty)}\right] - \frac{1}{k+j\omega}\left(e^{(-k-j\omega)(\infty)} - e^{(-k-j\omega)(0)}\right.\right.$$

$$= k\left[\frac{1}{k-j\omega}[1-0] - \frac{1}{k+j\omega}\left[e^{-(k+j\omega)(\infty)} - 1\right]\right]$$

$$= k\left[\frac{1}{k-j\omega} - \frac{1}{k+j\omega}[0-1]\right]$$

$$= k\left[\frac{1}{k-j\omega} + \frac{1}{k+j\omega}\right]$$

$$= k\left[\frac{k+j\omega+k-j\omega}{k^2+\omega^2}\right]$$

$$= k\left(\frac{2k}{k^2+\omega^2}\right)$$

$$= \frac{2k^2}{k^2+\omega^2}$$

$$= \frac{2k^2}{k^2\left(1+\frac{\omega^2}{k^2}\right)}$$

$$= \frac{2}{1+\frac{\omega^2}{k^2}}$$

$$\therefore S_{XY}(\omega) = \frac{2}{1+\left(\frac{\omega}{k}\right)^2}$$

5. A random process is given by $w(t) = x(t) + y(t)$. If $x(t)$ and $y(t)$ are jointly WSS r.p's, then find (i) ACF and (ii) PSD fn of $w(t)$ for the following cases.

(a) $x(t)$ and $y(t)$ are correlated

(b) $x(t)$ and $y(t)$ are uncorrelated

(c) $x(t)$ and $y(t)$ are uncorrelated with zero mean.

Sol: Given $w(t) = x(t) + y(t)$

(a) If $x(t)$ and $y(t)$ are correlated

The ACF of $w(t) = R_{ww}(\tau) = E[w(t)\, w(t+\tau)]$

$$= E[(x(t)+y(t))\,(x(t+\tau)+y(t+\tau))]$$

$$= E[x(t)\,x(t+\tau) + x(t)\,y(t+\tau) + y(t)\,x(t+\tau) + y(t)\,y(t+\tau)]$$

$$= \underbrace{E[x(t)\,x(t+\tau)]}_{ACF} + \underbrace{E[x(t)\,y(t+\tau)]}_{CCF} + \underbrace{E[y(t)\,x(t+\tau)]}_{CCF} + \underbrace{E[y(t)\,y(t+\tau)]}_{ACF}$$

$$= R_{XX}(\tau) + R_{XY}(\tau) + R_{YX}(\tau) + R_{YY}(\tau)$$

$$\boxed{\therefore R_{ww}(\tau) = R_{xx}(\tau) + R_{xy}(\tau) + R_{yx}(\tau) + R_{yy}(\tau)}$$

Apply fourier transform on both sides of above eqn we get,

$$F[R_{ww}(\tau)] = F[R_{xx}(\tau) + R_{xy}(\tau) + R_{yx}(\tau) + R_{yy}(\tau)]$$

By linearity prop. of F.T., we get

$$F[R_{ww}(\tau)] = F[R_{xx}(\tau)] + F[R_{xy}(\tau)] + F[R_{yx}(\tau)] + F[R_{yy}(\tau)]$$

We know $R_{xx}(\tau) \xleftrightarrow{F.T} S_{xx}(\omega)$ = The PSD of $x(t)$

$R_{yy}(\tau) \xleftrightarrow{F.T} S_{yy}(\omega)$ $\quad R_{ww}(\tau) \xleftrightarrow{F.T} S_{ww}(\omega)$

$R_{xy}(\tau) \xleftrightarrow{F.T} S_{xy}(\omega)$

$R_{yx}(\tau) \xleftrightarrow{F.T} S_{yx}(\omega)$

$$\boxed{\text{PSD of } w(t),\ S_{WW}(\omega) = S_{XX}(\omega) + S_{XY}(\omega) + S_{YX}(\omega) + S_{YY}(\omega)}$$

(b) If $x(t)$ and $y(t)$ are uncorrelated,

let $R_{XX}(\tau) = E[X(t)\, X(t+\tau)]$

$R_{XY}(\tau) = E[X(t)\, Y(t+\tau)] = E[X(t)]\, E[Y(t+\tau)] = \bar{X}\,\bar{Y}$

$R_{YX}(\tau) = E[Y(t)\, X(t+\tau)] = E[Y(t)]\, E[X(t+\tau)] = \bar{Y}\,\bar{X}$

$R_{YY}(\tau) = E[Y(t)\, Y(t+\tau)]$

The ACF of $w(t)$ $R_{WW}(\tau) = R_{XX}(\tau) + \bar{X}\cdot\bar{Y} + \bar{Y}\bar{X} + R_{YY}(\tau)$

$$\boxed{R_{WW}(\tau) = R_{XX}(\tau) + R_{YY}(\tau) + 2\cdot\bar{X}\,\bar{Y}}$$

For uncorrelated condition,

$S_{XY}(\omega) = 2\pi\bar{X}\,\bar{Y}\,\delta(\omega)$

$S_{YX}(\omega) = 2\pi\,\bar{X}\,\bar{Y}\,\delta(\omega)$

$$\therefore \boxed{S_{WW}(\omega) = S_{XX}(\omega) + S_{YY}(\omega) + 4\pi\bar{X}\,\bar{Y}\,\delta(\omega)}$$

(c) If $x(t)$ and $y(t)$ are uncorrelated with zero mean.

Here $\bar{X} = 0$, $\bar{Y} = 0$

The ACF of $w(t)$ $R_{WW}(\tau) = R_{XX}(\tau) + R_{YY}(\tau) + 2(0)(0)$

$$\boxed{R_{WW}(\tau) = R_{XX}(\tau) + R_{YY}(\tau)}$$

The PSD of $w(t) = S_{WW}(\tau)$

$S_{WW}(\omega) = S_{XX}(\omega) + S_{YY}(\omega) + 4\pi(0)(0)\delta(\omega)$

$$\boxed{S_{WW}(\omega) = S_{XX}(\omega) + S_{YY}(\omega)}$$

6. Let the R.P $w(t) = A x(t) + B y(t)$. Here $x(t)$ and $y(t)$ are jointly WSS random processes (i) Find power spectrum density $S_{ww}(\omega)$ ~~if x(t) and y(t)~~

(ii) PSD $S_{ww}(\omega)$ if $x(t)$ and $y(t)$ are uncorrelated random processes.

(iii) PSD $S_{ww}(\omega)$ if $x(t)$ and $y(t)$ are uncorrelated with zero mean values.

(iv) Find cross PSD's of $S_{xw}(\omega)$ & $S_{yw}(\omega)$.

Sol: Given $w(t) = A\, x(t) + B\, y(t)$

(i) The ACF of $w(t)$ = $R_{ww}(\tau) = E[w(t)\, w(t+\tau)]$

$$= E[(A\, x(t) + B\, y(t))\,(A\, x(t+\tau) + B\, y(t+\tau))]$$

$$= E[A^2 x(t)\, x(t+\tau) + AB\, x(t) y(t+\tau) + BA\, y(t) x(t+\tau) + B^2 y(t)\, y(t+\tau)]$$

$$= A^2 \underbrace{E[x(t)\, x(t+\tau)]}_{ACF} + AB \underbrace{E[x(t)\, y(t+\tau)]}_{CCF} + AB \underbrace{E[y(t)\, x(t+\tau)]}_{CCF} + B^2 \underbrace{E[y(t) y(t+\tau)]}_{ACF}$$

$$R_{ww}(\tau) = A^2 R_{xx}(\tau) + AB\, R_{xy}(\tau) + AB\, R_{yx}(\tau) + B^2 R_{yy}(\tau)$$

Apply fourier transform on both sides of the above equation, we get

$$F[R_{ww}(\tau)] = F[A^2 R_{xx}(\tau) + AB\, R_{xy}(\tau) + AB\, R_{yx}(\tau) + B^2 R_{yy}(\tau)]$$

By linearity property of F.T, we get

$$= A^2 F[R_{xx}(\tau)] + AB\, F[R_{xy}(\tau)] + AB\, F[R_{yx}(\tau)] + B^2 F[R_{yy}(\tau)]$$

We know that $R_{xx}(\tau) \xleftrightarrow{F.T} S_{xx}(\omega)$

Similarly the others

The PSD of $w(t) = S_{WW}(\omega) = A^2 S_{XX}(\omega) + AB\, S_{XY}(\omega) + AB\, S_{YX}(\omega) + B^2 S_{YY}(\omega)$

$$\boxed{\therefore S_{WW}(\omega) = A^2 S_{XX}(\omega) + AB\, S_{XY}(\omega) + AB\, S_{YX}(\omega) + B^2 S_{YY}(\omega)}$$

(ii) if $x(t)$ and $y(t)$ are uncorrelated random processes.

then $R_{XX}(\tau) = A^2 E[x(t)\, x(t+\tau)]$

$R_{XY}(\tau) = AB\, E[x(t)\, y(t+\tau)] = AB\, \bar{X} \cdot \bar{Y}$

$R_{YX}(\tau) = AB\, E[y(t)\, x(t+\tau)] = AB\, \bar{X} \cdot \bar{Y}$

$R_{YY}(\tau) = B^2\, E[y(t)\, y(t+\tau)] = B^2 E[y(t)\, y(t+\tau$

The ACF of $R_{WW}(\tau) = A^2 R_{XX}(\tau) + AB\, \bar{X} \cdot \bar{Y} + AB\, \bar{X} \cdot \bar{Y} + B^2 R_{YY}(\tau)$

$$R_{WW}(\tau) = A^2 R_{XX}(\tau) + B^2 R_{YY}(\tau) + 2AB\bar{X}$$

We know

$S_{XY}(\omega) = 2\pi\, \bar{X} \cdot \bar{Y}\, \delta(\omega)$

$S_{YX}(\omega) = 2\pi\, \bar{X} \cdot \bar{Y}\, \delta(\omega)$

$= A^2 S_{XX}(\omega) + B^2 S_{YY}(\omega) + 4\pi\, \bar{X}\bar{Y}\, AB\, \delta(\omega)$

$$\boxed{S_{WW}(\omega) = A^2 S_{XX}(\omega) + B^2 S_{YY}(\omega) + 4\pi AB\, \bar{X}\, \bar{Y}\, \delta(\omega)}$$

(iii) if $x(t)$ and $y(t)$ are uncorrelated with zero mean values i.e. $\bar{X} = 0$, $\bar{Y} = 0$

$S_{WW}(\omega) = A^2 S_{XX}(\omega) + B^2 S_{YY}(\omega) + 4\pi AB(0)(0)\delta(\omega)$

$$\boxed{S_{WW}(\omega) = A^2 S_{XX}(\omega) + B^2 S_{YY}(\omega)}$$

(iv) The CCF b/n $x(t)$ and $w(t) = R_{xw}(\omega) = E[x(t) \cdot w(t+\tau)]$.

$$= E[x(t)]\ E[w(t+\tau)]$$

$$= E[x(t)]\ E[A\ x(t+\tau) + B\ y(t+\tau)]$$

$$= A E[x(t)\, x(t+\tau)] + B\ E[x(t)\, y(t+\tau)]$$

$$R_{xw}(\omega) = A R_{xx}(\tau) + B R_{xy}(\tau)$$

By fourier transform applification, we have

$$\boxed{S_{xw}(\omega) = A S_{xx}(\omega) + B S_{xy}(\omega)}$$

The CCF b/n $y(t)$ and $w(t) = R_{yw}(\omega) = E[y(t) \cdot w(t+\tau)]$

$$= E[y(t)]\ E[w(t+\tau)]$$

$$= E[y(t)]\ E[A\ x(t+\tau) + B\, y(t+\tau)]$$

$$= A E[x(t+\tau)\, y(t)] + B E[y(t)\, y(t+\tau)]$$

$$R_{yw}(\omega) = A R_{yx}(\tau) + B R_{yy}(\tau)$$

By fourier transform application, we have

$$\boxed{S_{yw}(\omega) = A S_{yx}(\omega) + B S_{yy}(\omega)}$$

7. Find ACF, average power and RMS band width for the following PSD.

(i) $S_{xx}(\omega) = \begin{cases} A\cos\left(\dfrac{\pi\omega}{2W}\right) & ; |\omega| \le W \\ 0 & ; \text{else where} \end{cases}$

(ii) $S_{xy}(\omega) = \begin{cases} 1 + \dfrac{j\omega}{k} & ; |W| \le k \\ 0 & ; \text{elsewhere.} \end{cases}$

(iii) $S_{XX}(\omega) = \begin{cases} 4 - \frac{\omega^2}{9} & ; |\omega| \le 6 \\ 0 & ; \text{elsewhere} \end{cases}$

(iv) if $R_{XX}(\tau) = P\cos^4(\omega_0 \tau)$ then find $S_{XX}(\omega)$, P_{XX} & W_{rms}

Sol: i) Given $S_{XX}(\omega) = \begin{cases} A\cos\left(\frac{\pi\omega}{2W}\right) & ; |\omega| \le W \\ 0 & ; \text{elsewhere} \end{cases}$

The ACF of $x(t) = R_{XX}(\tau) = F^{-1}[S_{XX}(\omega)]$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, e^{j\omega\tau}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-W}^{W} A\cos\left(\frac{\pi\omega}{2W}\right) e^{j\omega\tau}\, d\omega \qquad \because \int e^{at}\cos bt\, dt = \frac{e^{at}}{a^2+b^2}(a\cos bt + b\sin bt)$$

$$= \frac{A}{2\pi}\int_{-W}^{W} e^{(j\tau)\omega} \cos\left(\frac{\pi}{2W}\right)\omega\, d\omega$$

$$= \frac{A}{2\pi}\left[\frac{e^{j\tau\omega}}{(j\tau)^2+\left(\frac{\pi}{2W}\right)^2}\left[(j\tau)\cos\left(\frac{\pi}{2W}\omega\right) + \frac{\pi}{2W}\sin\left(\frac{\pi\omega}{2W}\right)\right]\right]_{-W}^{W}$$

$$= \frac{A}{2\pi}\left(\frac{e^{jw\tau}}{(j\tau)^2+\left(\frac{\pi}{2W}\right)^2}\left[j\tau\cos\left(\frac{\pi}{2W}\cdot W\right) + \frac{\pi}{2W}\sin\left(\frac{\pi}{2W}\cdot W\right)\right]\right.$$

$$\left. - \frac{e^{-j\tau w}}{-\tau^2+\frac{\pi^2}{4W^2}}\left[j\tau\cos\left(\frac{\pi}{2W}(-W)\right) + \frac{\pi}{2W}\left(\sin\frac{\pi}{2W}(-W)\right)\right]\right)$$

$$= \frac{A}{2\pi}\cdot\frac{1}{\left(\frac{\pi^2-4\omega^2\tau^2}{4W^2}\right)}\left[e^{j\omega\tau}\left(0+\frac{\pi}{2W}\right) - e^{-j\omega\tau}\left(0+\frac{\pi}{2W}(-1)\right)\right]$$

$$\therefore R_{XX}(\tau) = \frac{2AW^2}{\pi(\pi^2-4\omega^2\tau^2)}\left[e^{j\omega\tau}\left(\frac{\pi}{2W}\right) + e^{-j\omega\tau}\left(\frac{\pi}{2W}\right)\right]$$

$$= \frac{2AW^2}{\pi(\pi^2-4W^2\tau^2)} \times \frac{\pi}{2W}\left(e^{j\omega\tau} + e^{-j\omega\tau}\right)$$

$$= \frac{\cancel{2} A W^{\cancel{2}}}{\cancel{\pi}(\pi^2 - 4W^2\tau^2)} \times \frac{\cancel{\pi}}{\cancel{2W}} (2\cos W\tau)$$

$$\therefore R_{xx}(\tau) = \frac{2AW \cos W\tau}{\pi^2 - 4W^2\tau^2} \longrightarrow ①$$

Average power $= P_{xx} = R_{xx}(0)$

$\{\because \tau = 0 ; \quad \cos 0° = 1\}$

$$① \Rightarrow R_{xx}(0) = \frac{2AW}{\pi^2 - 4(0)} \cos 0°$$

$$R_{xx}(0) = \frac{2AW}{\pi^2} \text{ watts}$$

RMS bandwidth =

$$\omega_{rms}^2 = \frac{\int_{-\infty}^{\infty} \omega^2 S_{xx}(\omega)\, d\omega}{\int_{-\infty}^{\infty} S_{xx}(\omega)\, d\omega}$$

$$\int_{-\infty}^{\infty} \omega^2 S_{xx}(\omega)\, d\omega = \int_{-W}^{W} \omega^2 \, A\cos\left(\frac{\pi\omega}{2W}\right) d\omega$$

$$= A \int_{-W}^{W} \underbrace{\omega^2}_{u} \underbrace{\cos\left(\frac{\pi\omega}{2W}\right)}_{v} d\omega$$

$$= A \left[\omega^2 \frac{\sin\left(\frac{\pi\omega}{2W}\right)}{\frac{\pi\omega}{2W}} \Bigg|_{-W}^{W} - \int_{-W}^{W} 2\omega \cdot \frac{\sin \frac{\pi\omega}{2W}}{\frac{\pi}{2W}} d\omega \right]$$

$$= A \left[\left[\omega^2 \frac{\sin \frac{\pi W}{2W}}{\left(\frac{\pi}{2W}\right)}\right] - \left[W^2 \frac{\sin\left(\frac{-\pi W}{2W}\right)}{\left(\frac{\pi}{2W}\right)}\right] - 2\int_{-W}^{W} \underset{u}{\omega} \cdot \underset{v}{\frac{\sin\left(\frac{\pi\omega}{2W}\right)}{\frac{\pi}{2W}}} d\omega \right]$$

$$= A\left[\left[\frac{w^2(1)}{\pi/2w}\right] + \left[\frac{w^2(1)}{\frac{\pi}{2w}}\right] + 2\left[w\left(\frac{\cos\frac{\pi w}{2W}}{\left(\frac{\pi}{2w}\right)\left(\frac{\pi}{2w}\right)}\right)\Bigg|_{-w}^{w} - \int_{-w}^{w} - \frac{\cos\left(\frac{\pi w}{2W}\right)}{\frac{\pi}{2w}\cdot\frac{\pi}{2w}}\, dw\right]\right]$$

$$= A\left[\frac{2w^3}{\pi} + \frac{2w^3}{\pi} - 2\left[\omega\frac{\left(\cos\frac{\pi w}{2w}\right)}{\frac{\pi^2}{4w^2}} + w\frac{\cos\left(\frac{\pi w}{2w}\right)}{\pi^2/4w^2} + \frac{\sin\left(\frac{\pi w}{2W}\right)}{\frac{\pi^3}{8w^3}}\right]\Bigg|_{-w}^{w}\right]$$

$$= A\left(\frac{4\omega^3}{\pi} - 2\left[0+0+\frac{\left(\sin\left(\frac{\pi w}{2w}\right)\right)}{\pi^3/8w^3} + \frac{\sin\left(\frac{\pi w}{2W}\right)}{\pi^3/8w^3}\right]\right)$$

$$= A\left(\frac{4\omega^3}{\pi} + 2(-1)\left(\frac{8w^3}{\pi^3}\right) + 2(-1)\frac{8w^3}{\pi^3}\right)$$

$$= A\left[\frac{4\omega^3}{\pi} - \frac{16w^3}{\pi^3} - \frac{16w^3}{\pi^3}\right]$$

$$= A\left[\frac{4w^3}{\pi} - \frac{32w^3}{\pi^3}\right]$$

$$= \frac{4Aw^3}{\pi}\left(1 - \frac{8}{\pi^2}\right)$$

$$\int_{-\infty}^{\infty} S_{XX}(\omega)\,d\omega = 2\pi P_{XX} = 2\pi \times \frac{2Aw}{\pi^2}$$

$$= \frac{4Aw}{\pi}$$

RMS bandwidth, $\omega_{rms}^2 = \dfrac{\frac{4\cancel{A}w^{\cancel{3}}}{\cancel{\pi}}\left(1-\frac{8}{\pi^2}\right)}{\frac{4\cancel{A}\cancel{w}}{\cancel{\pi}}}$

$$\omega_{rms}^2 = \omega^2\left(1-\frac{8}{\pi^2}\right)$$

$$\omega_{rms} = \omega\sqrt{\left(1-\frac{8}{\pi^2}\right)} \text{ rad/sec.}$$

(ii). Given $S_{xy}(\omega) = \begin{cases} 1+\dfrac{j\omega}{k} & ; |\omega| \le k \\ 0 & ; \text{elsewhere} \end{cases}$

Sol: $R_{xy}(\tau) = F^{-1}[S_{xy}(\omega)] = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} S_{xy}(\omega)\, e^{j\omega\tau}\, d\omega$

$$= \frac{1}{2\pi}\int_{-k}^{k}\left(1+\frac{j\omega}{k}\right) e^{j\omega\tau}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-k}^{k} e^{j\omega\tau}\, d\omega + \frac{j}{2\pi k}\int_{-k}^{k} \omega e^{j\omega\tau}\, d\omega$$

$$= \frac{1}{2\pi}\left.\frac{e^{j\omega\tau}}{j\tau}\right|_{-k}^{k} + \frac{j}{2\pi k}\left[\omega\left(\frac{e^{j\omega\tau}}{j\tau}\right) - \frac{1 \cdot e^{j\omega\tau}}{(j\tau)^2}\right]_{-k}^{k}$$

$$= \frac{1}{\pi\tau}\left[\frac{e^{jk\tau}-e^{-jk\tau}}{2j}\right] + \frac{j}{2\pi k}\left[\left[\left(k\frac{e^{jk\tau}}{j\tau} - \frac{e^{jk\tau}}{-\tau^2}\right]\right. - \left\{k\frac{e^{-jk\tau}}{j\tau} - \frac{e^{-jk\tau}}{-\tau^2}\right]\right]$$

$$= \frac{1}{\pi\tau}\sin(k\tau) + \frac{j}{2\pi k}\cdot\frac{k}{j\tau}\left(e^{jk\tau}+e^{-jk\tau}\right) + \frac{j}{2\pi k}\cdot\frac{1}{\tau^2}\left(e^{jk\tau}-e^{-jk\tau}\right)$$

$$= \frac{1}{\pi\tau}\sin(k\tau) + \frac{1}{2\pi\tau}\, 2\cos k\tau + \frac{j}{2\pi k\tau^2}\left(2j\sin(k\tau)\right)$$

$$= \left(\frac{1}{\pi\tau} - \frac{1}{\pi k\tau^2}\right)\sin(k\tau) + \frac{1}{\pi\tau}\cos(k\tau)$$

$$\therefore R_{xy}(\tau) = \frac{1}{\pi\tau}\left(1-\frac{1}{k\tau}\right)\sin k\tau + \frac{1}{\pi\tau}\cos k\tau.$$

Average power, $P_{xy} = \frac{1}{2\pi}\int_{-\infty} S_{xy}(\omega)\,d\omega$

$$= \frac{1}{2\pi}\int_{-k}^{k}\left(1+\frac{j\omega}{k}\right)d\omega$$

$$= \frac{1}{2\pi}\int_{-k}^{k} 1\cdot d\omega + \frac{j}{2\pi k}\int_{-k}^{k}\omega\,d\omega$$

$$= \frac{1}{2\pi}\left[\omega\right]_{-k}^{k} + \frac{j}{2\pi k}\left[\frac{\omega^2}{2}\right]_{-k}^{k}$$

$$= \frac{2k}{2\pi} + \frac{j}{2\pi k}\cdot\frac{1}{2}\left[k^2-k^2\right]$$

$$\therefore P_{xy} = \frac{k}{\pi} \text{ watts.}$$

RMS bandwidth

Base band process $\quad \omega_{rms}^2 = \dfrac{\int_{-\infty}^{\infty}\omega^2 S_{xy}(\omega)\,d\omega}{\int_{-\infty}^{\infty} S_{xy}(\omega)\,d\omega}$

$$\int_{-\infty}^{\infty}\omega^2 S_{xy}(\omega)\,d\omega = \int_{-\infty}^{\infty}\omega^2\left(1+\frac{j\omega}{k}\right)d\omega$$

$$= \int_{-k}^{k}\omega^2 d\omega + \frac{j}{k}\int_{-k}^{k}\omega^3\,d\omega$$

$$= \left[\frac{\omega^3}{3}\right]_{-k}^{k} + \frac{j}{k}\left[\frac{\omega^4}{4}\right]_{-k}^{k}$$

$$= \left(\frac{k^3+k^3}{3}\right) + \frac{j}{k}\left[\frac{k^4}{4}-\frac{k^4}{4}\right]$$

$$= \frac{2k^3}{3}$$

$$\int_{-\infty}^{\infty} S_{xy}(\omega)\,d\omega = 2\pi P_{xy} = 2\pi\cdot\frac{k}{\pi} = 2k$$

$$\Rightarrow \omega^2_{rms} = \frac{2k^3}{3} \times \frac{1}{2k}$$

$$\omega^2_{rms} = \frac{k^2}{3}$$

$$\omega_{rms} = \frac{k}{\sqrt{3}} \text{ rad/sec.}$$

(iv) $R_{xx}(\tau) = P\cos^4(\omega_0\tau)$

$$R_{xx}(\tau) = P\left[\frac{3}{8} + \frac{1}{2}\cos(2\omega_0\tau) + \frac{1}{8}\cos(4\omega_0\tau)\right]$$

$$\because \cos^4\theta = (\cos^2\theta)^2 = \left(\frac{1+\cos 2\theta}{2}\right)^2$$

$$= \frac{1 + \cos^2 2\theta + 2\cos 2\theta}{4}$$

$$= \frac{1}{4} + \frac{\left(\frac{1+\cos 4\theta}{2}\right)}{4} + \frac{1}{2}\cos 2\theta$$

$$= \frac{1}{4} + \frac{1}{8} + \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta$$

$$\therefore R_{xy}(\tau) = \frac{3}{8} + \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta.$$

Apply F.T. on both sides, we get

$$F[R_{xx}(\tau)] = F\left[\frac{3P}{8} + \frac{P}{2}\cos(2\omega_0\tau) + \frac{P}{8}\cos(4\omega_0\tau)\right].$$

By linearity property, we have:

$$S_{xx}(\omega) = \frac{3P}{8}F[1] + \frac{P}{2}F[\cos(2\omega_0\tau)] + \frac{P}{8}F[\cos(4\omega_0\tau)]$$

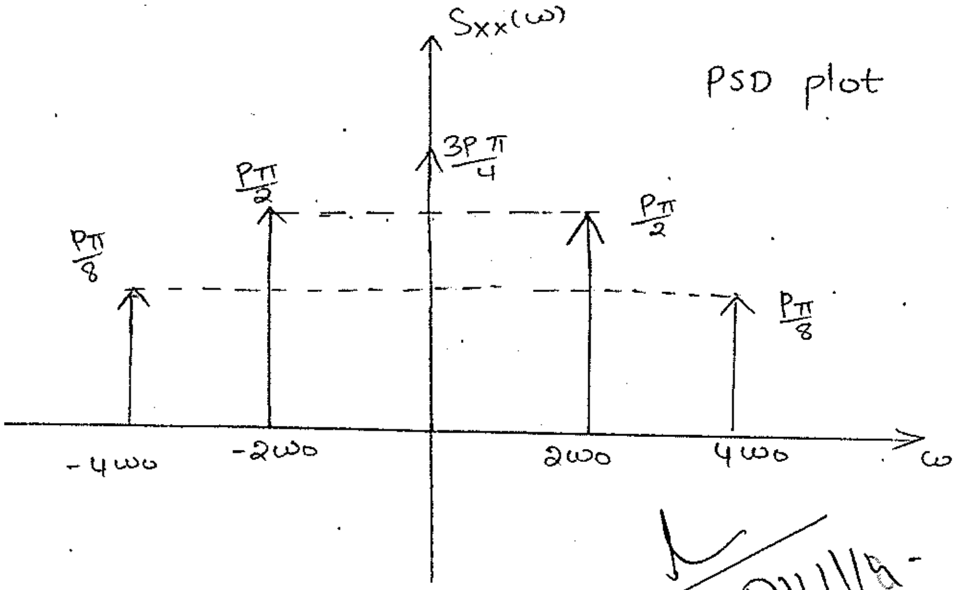
We know $1 \xleftrightarrow{F.T} 2\pi\,\delta(\omega)$

$$\cos\omega_0\tau \xleftrightarrow{F.T} \pi[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]$$

$$\cos(2\omega_0\tau) \xleftrightarrow{F.T} \pi[\delta(\omega-2\omega_0) + \delta(\omega+2\omega_0)]$$

$$S_{xx}(\omega) = \frac{3P}{8}\cdot 2\pi\,\delta(\omega) + \frac{P}{2}\cdot\pi[\delta(\omega-2\omega_0) + \delta(\omega+2\omega_0)]$$
$$+ \frac{P}{8}\pi[\delta(\omega-4\omega_0) + \delta(\omega+4\omega_0)]$$

$$S_{xx}(\omega) = \frac{3P}{4}\pi\,\delta(\omega) + \frac{P\pi}{2}\left[\delta(\omega-2\omega_0)+\delta(\omega+2\omega_0)\right] + \frac{P\pi}{8}\left[\delta(\omega-4\omega_0) + \delta(\omega+\right.$$



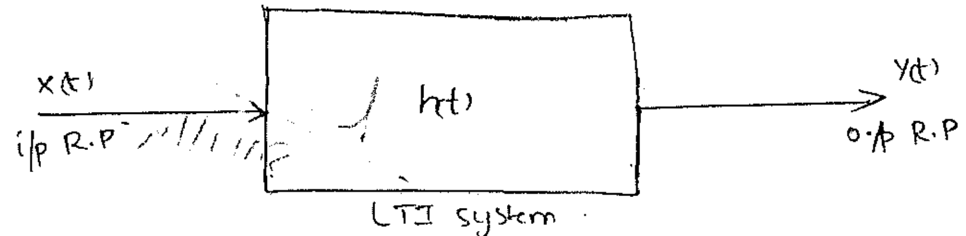
UNIT - VI

5/11/14 * UNIT-5 *

* Response of Linear Systems To Random Signals *

→ Output Response of Linear System or Response of Linear Time Invariant system (LTI):

Let a random process $X(t)$ is applied to an LTI system whose impulse response is $h(t)$, as shown in figure



$Y(t)$ be the output random process. Then the o/p response of the linear system is defined by

$$\boxed{\text{o/p response of LTI system} = Y(t) = h(t) \circledast X(t) = \int_{-\infty}^{\infty} h(\tau)\, X(t-\tau)\, d\tau}$$

or

$$Y(t) = X(t) \circledast h(t) = \int_{-\infty}^{\infty} X(\tau)\, h(t-\tau)\, d\tau$$

Here $X(t)$ is i/p random process, $h(t)$ is impulse response of LTI system and $Y(t)$ is o/p random process.

→ Mean value of o/p response:

The o/p response of LTI system is defined by

$$Y(t) = h(t) * X(t) = \int_{-\infty}^{\infty} h(\tau)\, X(t-\tau)\, d\tau$$

Apply mean (expectation) operator, we get,

$$E[Y(t)] = E[h(t) * X(t)]$$

$$= E\left[\int_{-\infty}^{\infty} h(\tau)\, X(t-\tau)\, d\tau\right]$$

$$= \int_{-\infty}^{\infty} h(\tau)\, E[x(t-\tau)]\, d\tau$$

$\because x(t)$ is a WSS R.P. then $E[x(t)] = E[x(t+\tau)]$

$$= \overline{x} = \text{constant.}$$

$$E[Y(t)] = \int_{-\infty}^{\infty} h(\tau)\, \overline{x} \cdot d\tau$$

$$E[Y(t)] = \overline{x} \int_{-\infty}^{\infty} h(\tau) \cdot d\tau$$

We know frequency response of LTI system $= H(\omega)$

$$= F[h(\tau)] = \int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega\tau}\, d\tau$$

Zero frequency response of LTI system $= H(0) = \int_{-\infty}^{\infty} h(\tau)\, e^{-j(0)\tau} d\tau$

$$\Rightarrow H(0) = \int_{-\infty}^{\infty} h(\tau)\, d\tau$$

So now, $E[Y(t)] = \overline{x} \int_{-\infty}^{\infty} h(\tau)\, d(\tau)$

$$\overline{Y} = \overline{x} \cdot H(0)$$

$\therefore$ $\boxed{\overline{Y} = E[Y(t)] = H(0) \cdot \overline{x}}$ = The mean value of o/p respon

Thus, the mean value of o/p response is equal to th product of the mean value of i/p random process and zero frequency-response of the LTI system.

$\Rightarrow$ Mean square value of o/p response:

The o/p response of LTI system is defined by

$$Y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau = \int_{-\infty}^{\infty} h(\tau_1)\, x(t-\tau_1)\, d\tau_1 = \int_{-\infty}^{\infty} h(\tau_2)\, x(t-\tau_2)\, d\tau_2$$

The meansquare value of o/p response $= E[Y^2(t)]$

$$= E[Y(t) \cdot Y(t)]$$

$$= E[(h(t) * x(t))\,(h(t) * x(t))]$$

$$= E\left[\left(\int_{-\infty}^{\infty} h(\tau_1) x(t-\tau_1)\, d\tau_1\right)\left(\int_{-\infty}^{\infty} h(\tau_2) x(t-\tau_2)\, d\tau_2\right)\right]$$

$$= E\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t-\tau_1)\, x(t-\tau_2)\, h(\tau_1)\, h(\tau_2)\, d\tau_1\, d\tau_2\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E\left[x(t-\tau_1)\, x(t-\tau_2)\right] h(\tau_1)\, h(\tau_2)\, d\tau_1 d\tau_2$$

We know that

$$E[x(t-\tau_1)\, x(t-\tau_2)] = R_{xx}\left[(\cancel{t}-\tau_2)-(\cancel{t}-\tau_1)\right] = R_{xx}(\tau_1-\tau_2)$$

$\because$ For WSS R.P, $R_{xx}(t_1, t_2) = E[x(t_1)x(t_2)] = R_{xx}(t_2-t_1)$

$$\boxed{\therefore E[Y^2(t)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{xx}(\tau_1-\tau_2)\, h(\tau_1)\, h(\tau_2)\, d\tau_1\, d\tau_2}$$

= The meansquare value of o/p response.

The meansquare value of o/p response is a function of 'τ' only, not absolute time t.

*Types of Random Processes:

There are four types of random processes.

(a) Low pass random processes

(b) Band pass random processes.

(c) Band limited random processes

(d) Narrow Band random processes.

(a) Low Pass Random Processes:

A randomprocess is defined as the low pass random process $X(t)$, if its power spectral density $S_{xx}(\omega)$ has significant components

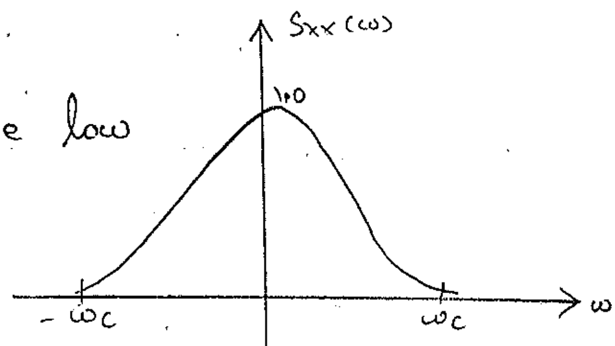


Fig: Power Spectrum of LPRP

within the frequency band as shown in figure. For example base band signals, such as speech, image and video are low pass random processes.

(b) Band Pass Random Processes :

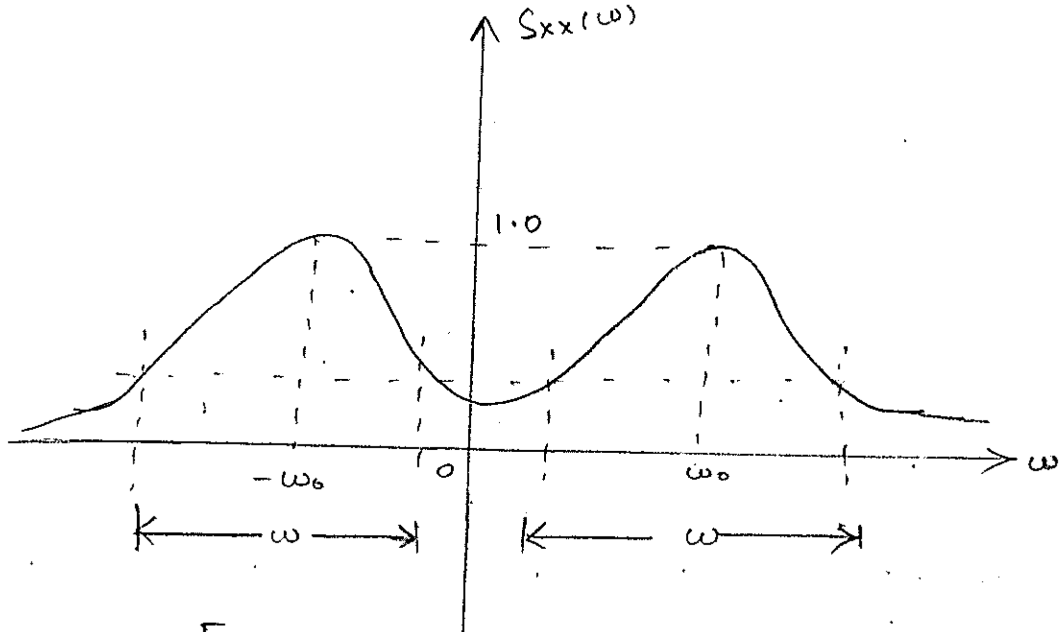


Fig: Power spectrum of BPRP

A random process X(t) is called a band pass random process if its power spectral density $S_{XX}(\omega)$ has its significant component within a bandwidth W that doesnot include $\omega=0$, (as shown in figure). But in practise, the spectrum may have a small amount of power spectrum at $\omega=0$, as shown in figure. The spectral components outside the band W are very small and can be neglected. For example, modulated signals with carrier frequency ω_0 and bandwidth W are band pass random processes.

The noise transmitting over a communication channel can be modelled as a band pass random process.

(C) Band Limited Random Processes:

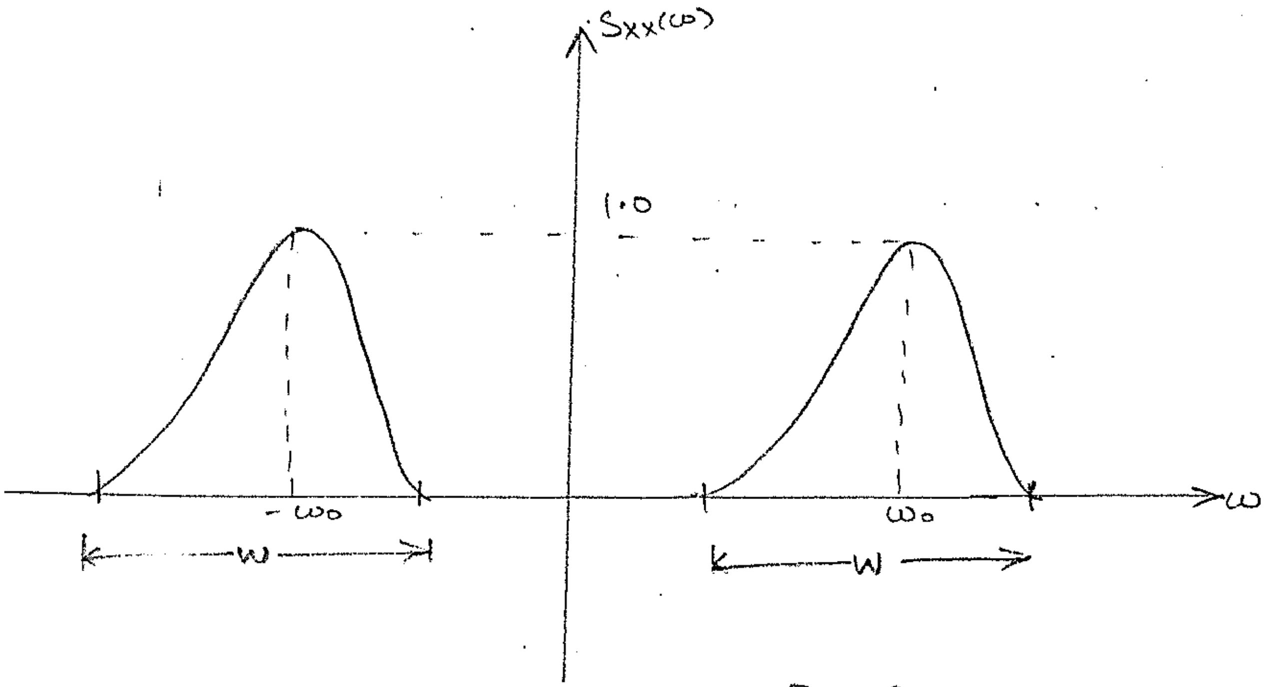


Fig: Power spectrum of BLRP

A band pass random process is said to be band limited random process if its power spectrum components are zero outside the frequency band of width W that doesnot include $\omega = 0$. The power spectrum density of Band limited random process is as shown in fig.

(d) Narrow Band Random Processes:

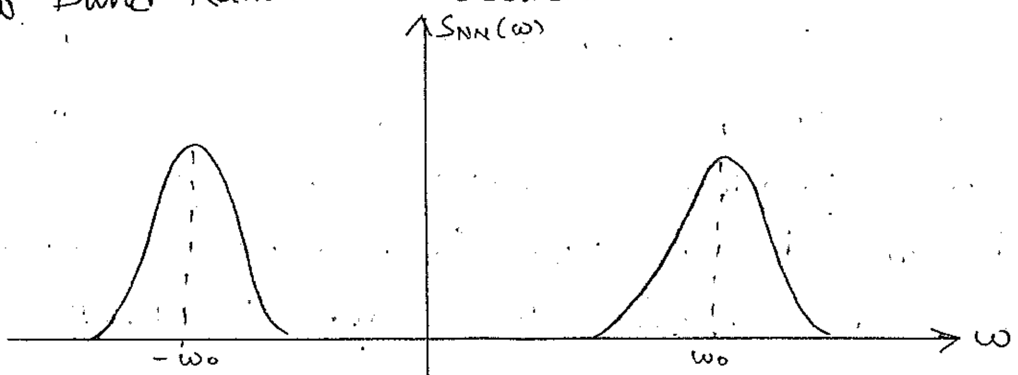
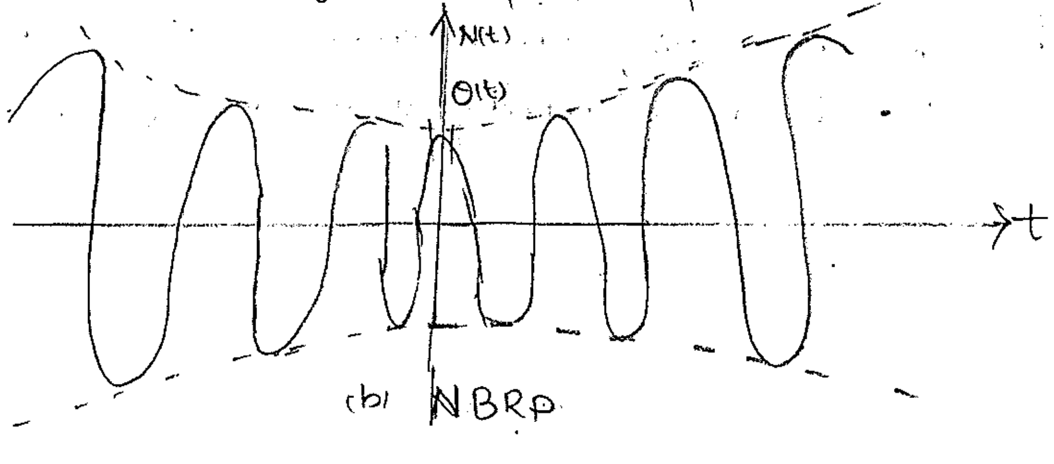


Fig: (a) Power spectrum of NBRP



(b) NBRP

A band limited random process is said to be a narrow band process if the bandwidth W is very small, compared to the band central frequency, i.e. $W << W_0$ where W = bandwidth and w_0 is the frequency at which the power spectrum is maximum.

The power density of a narrow band random process $N(t)$ is as shown in fig(a).

The narrow band process can be modelled as a Cosine function slowly varying in amplitude and phase with frequency w_0 as shown in fig(b). It can be expressed as

$$N(t) = A(t) \cos[w_0 t + \theta(t)]$$

where $A(t)$ - amplitude of R.P.

$\theta(t)$ - phase of R.P.

* Representation of Narrow Band Random Process:

For any arbitrary WSS random process $N(t)$

$$N(t) = A(t) \cos[w_0 t + \theta(t)]$$

$$= A(t) [\cos w_0 t \cos\theta(t) - \sin w_0 t \sin\theta(t)].$$

$$N(t) = A(t) \cos(\theta(t)) \cos(w_0 t) - A(t) \sin\theta(t) \sin w_0 t \rightarrow (1)$$

The Quadrature form of Narrowband process is defined by

$$N(t) = X(t) \cos(w_0 t) - Y(t) \sin(w_0 t) \longrightarrow (2)$$

Comparing (1) and (2), we have

The inphase component of $N(t) = X(t) = A(t) \cos[\theta(t)]$

The Quadrature phase component of $N(t) = Y(t) = A(t) \sin[\theta(t)]$

The relationship b/n the processes $A(t)$ and $\theta(t)$ are given by

$$x^2(t) + y^2(t) = A^2(t)\cos^2[\theta(t)] + A^2(t)\sin^2[\theta(t)]$$

$$x^2(t) + y^2(t) = A^2(t)(\cos^2[\theta(t)] + \sin^2[\theta(t)])$$

$$= A^2(t) \cdot 1$$

$$\Rightarrow x^2(t) + y^2(t) = A^2(t)$$

$$\boxed{A(t) = \sqrt{x^2(t) + y^2(t)}}$$

$$\frac{y(t)}{x(t)} = \frac{A(t)\sin[\theta(t)]}{A(t)\cos[\theta(t)]}$$

$$\frac{y(t)}{x(t)} = \tan\theta[t]$$

$$\Rightarrow \boxed{\theta(t) = \tan^{-1}\left[\frac{y(t)}{x(t)}\right]}$$

Test * Properties of Band limited Random Process;

* Let $N(t)$ be any band limited WSS R.P. with zero mean value and power spectrum density $S_{NN}(\omega)$. If the random process is represented by $N(t) = x(t)\cos\omega_0 t - y(t)\sin\omega_0 t$ then some important properties of BLRP are given below.

1. If $N(t)$ is WSS; then $x(t)$ and $y(t)$ are jointly widesense stationary rp's
2. If $N(t)$ has zero mean i.e $E[N(t)] = 0$ then $E[x(t)] = E[y(t)] = 0$
3. The mean square values of the processes are equal i.e, $E[N^2(t)] = E[x^2(t)] = E[y^2(t)]$

4. Both processes $X(t)$ and $Y(t)$ have same autocorrelation functions i.e, $R_{XX}(\tau) = R_{YY}(\tau)$

5. The cross-correlation function of $X(t)$ and $Y(t)$ are given by $R_{XY}(\tau) = -R_{YX}(\tau)$.

   If the processes are orthogonal then

$$R_{XY}(\tau) = R_{YX}(\tau) = 0.$$

6. Both $X(t)$ and $Y(t)$ have same power spectral densities

$$S_{YY}(\omega) = S_{XX}(\omega) = \begin{cases} S_N(\omega-\omega_0) + S_N(\omega+\omega_0) & ; |\omega| \le W_0 \\ 0 & ; \text{otherwise} \end{cases}$$

7. The cross-power spectrums of r.p's $S_{XY}(\omega) = -S_{YX}(\omega)$.

8. If $N(t)$ is a Gaussian random process then $X(t)$ and $Y(t)$ are jointly Gaussian r.

9. The relationship b/n ACF and PSD $[S_{NN}(\omega)]$ is

$$R_{XX}(\tau) = \frac{1}{\pi}\int_0^\infty S_{NN}(\omega)\cos[(\omega-\omega_0)\tau]\,d\omega$$

$$R_{XY}(\tau) = \frac{1}{\pi}\int_0^\infty S_{NN}(\omega)\sin[(\omega-\omega_0)\tau]\,d\omega$$

10. If $N(t)$ is zero mean Gaussian and its PSD, $S_{NN}(\omega)$ is symmetric about $\pm\omega_0$, then $X(t)$ and $Y(t)$ are statistically independent.

* Cross Correlation Function of o/p Response:

(i) The CCF b/n $X(t)$ and $Y(t)$ = $R_{xy}(\tau) = E[X(t)\,Y(t+\tau)]$

o/p response of linear system $Y(t) = h(t) * X(t)$

$$= \int_{-\infty}^{\infty} h(\tau)\, X(t-\tau)\, d\tau$$

Put $\tau = \tau_1$, we get

$$= \int_{-\infty}^{\infty} h(\tau_1)\, X(t-\tau_1)\, d\tau_1$$

$$Y(t+\tau) = \int_{-\infty}^{\infty} h(\tau_1)\, X(t+\tau-\tau_1)\, d\tau_1$$

$$R_{xy}(\tau) = E\left[X(t) \int_{-\infty}^{\infty} h(\tau_1)\, X(t+\tau-\tau_1)\, d\tau_1\right]$$

$$= \cancel{E}\left[\int_{-\infty}^{\infty} h(\tau_1)\, E[X(t)\, X(t+\tau-\tau_1)] d\tau_1\right.$$

We know ~~that~~ If $X(t)$ is WSS then

$$R_{xx}(t_1, t_2) = E[X(t_1)\, X(t_2)] = R_{xx}(t_2 - t_1)$$

$$E[X(t) \cdot X(t+\tau-\tau_1)] = R_{xx}((t+\tau-\tau_1) - t)$$

$$= R_{xx}(\tau - \tau_1)$$

Now $R_{xy}(\tau) = \int_{-\infty}^{\infty} h(\tau_1)\, R_{xx}(\tau-\tau_1)\, d\tau_1$

$$\boxed{\therefore R_{xy}(\tau) = h(\tau) * R_{xx}(\tau)}$$

(ii) The CCF b/n $Y(t)$ and $X(t)$ = $R_{yx}(\tau) = E[Y(t)\, X(t+\tau)]$

o/p response of LTI $Y(t) = h(t) * X(t)$

$$= \int_{-\infty}^{\infty} h(\tau)\, X(t-\tau)\, d\tau$$

Put $\tau = \tau_1$

$$= \int_{-\infty}^{\infty} h(\tau_1)\, X(t-\tau_1)\, d\tau_1$$

$$R_{YX}(\tau) = E\left[\int_{-\infty}^{\infty} h(\tau_1)\, x(t-\tau_1)\, d\tau_1 \cdot x(t+\tau)\right]$$

$$= E\left[\int_{-\infty}^{\infty} h(\tau_1)\, x(t-\tau_1)\, x(t+\tau)\, d\tau_1\right]$$

$$= \int_{-\infty}^{\infty} h(\tau_1)\, E\left[x(t-\tau_1)\, x(t+\tau)\right] d\tau_1$$

We know that if $x(t)$ is WSS then

$$R_{XX}(t_1, t_2) = E[X(t_1) \cdot X(t_2)] = R_{XX}(t_2 - t_1)$$

$$R_{XX}(t-\tau_1, t+\tau) = E[X(t-\tau_1) X(t+\tau)] = R_{XX}[(t+\tau) - (t-\tau_1)]$$

$$= R_{XX}(\tau + \tau_1)$$

Now $$R_{XY}(\tau) = \int_{-\infty}^{\infty} h(\tau_1)\, R_{XX}(\tau + \tau_1)\, d\tau_1$$

Put $\tau_1 = -\tau_2 \Rightarrow d\tau_1 = -d\tau_2$

$$= \int_{\infty}^{-\infty} h(-\tau_2)\, R_{XX}(\tau + (-\tau_2)) - d\tau_2$$

$$= -\int_{-\infty}^{+\infty} h(-\tau_2)\, R_{XX}(\tau_1 - \tau_2) - d\tau_2$$

$$= \int_{-\infty}^{\infty} h(-\tau_2)\, R_{XX}(\tau_1 - \tau_2)\, d\tau_2$$

$$\boxed{\therefore R_{YX}(\tau) = h(-\tau) * R_{XX}(\tau)}$$

(iii) The ACF of $Y(t) = R_{YY}(\tau) = E[Y(t)\, Y(t+\tau)]$.

o/p response of LTI $Y(t) = h(t) * x(t)$

$$= \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau_1)\, X(t-\tau_1)\, d\tau_1$$

$$R_{YY}(\tau) = E\left[\int_{-\infty}^{\infty} h(\tau_1)\, X(t-\tau_1)\, d\tau_1 \quad Y(t+\tau)\right]$$

$$= E\left[\int_{-\infty}^{\infty} h(\tau_1)\; X(t-\tau_1)\; Y(t+\tau)\, d\tau_1\right]$$

$$= \int_{-\infty}^{\infty} h(\tau_1)\quad E\left[X(t-\tau_1) Y(t+\tau)\right] d\tau_1$$

We know if $X(t)$ & $Y(t)$ are jointly WSS then

$$R_{XY}(t_1, t_2) = E\left[X(t_1)\; Y(t_2)\right] = R_{XY}(t_2 - t_1)$$

$$E\left[X(t-\tau_1)\; Y(t+\tau)\right] = R_{XY}\left((t+\tau) - (t-\tau_1)\right) = R_{XY}(\tau + \tau_1)$$

$$\Rightarrow R_{YY}(\tau) = \int_{-\infty}^{\infty} h(\tau_1)\; R_{XY}\;(\tau + \tau_1)$$

Replace τ_1 by $-\tau_2 \Rightarrow d\tau_1 = -d\tau_2$

$$= \int_{+\infty}^{-\infty} h(-\tau_2)\; R_{XY}\left(\tau + (-\tau_2)\right)(-d\tau_2)$$

$$= -\int_{-\infty}^{\infty} h(-\tau_2)\; R_{XY}(\tau - \tau_2)\; (-d\tau_2)$$

$$= \int_{-\infty}^{\infty} h(-\tau_2)\quad R_{XY}(\tau - \tau_2)\; d\tau_2$$

$$\boxed{\therefore R_{YY}(\tau) = h(-\tau) * R_{XY}(\tau)}$$

(v) The ACF of $Y(t) = R_{YY}(\tau) = E[Y(t) \cdot Y(t+\tau)]$

o/p response of linear system $= Y(t) = h(t) * X(t)$

$$= \int_{-\infty}^{\infty} h(\tau_1)\, x(t-\tau)\, d\tau$$

$$\Rightarrow Y(t+\tau) = \int_{-\infty}^{\infty} h(\tau_1)\, x((t+\tau)-\tau_1)\, d\tau_1$$

$$R_{YY}(\tau) = E\left[Y(t) \int_{-\infty}^{\infty} h(\tau_1)\, x((t+\tau)-\tau_1)\, d\tau_1 \right]$$

$$= E\left[\int_{-\infty}^{\infty} h(\tau_1)\, Y(t)\, X((t+\tau)-\tau_1)\, d\tau_1 \right]$$

$$= \int_{-\infty}^{\infty} h(\tau_1)\, E\left[Y(t)\, X(t+\tau-\tau_1) \right] d\tau_1$$

We know, If $x(t)$ and $Y(t)$ are jointly WSS then

$$R_{YX}(t_1,t_2) = E[Y(t_1)\, X(t_2)] = R_{YX}(t_2-t_1)$$

$$E[Y(t)\, X(t+\tau-\tau_1)] = R_{YX}((t+\tau-\tau_1)-t) = R_{YX}(\tau-\tau_1)$$

$$\Rightarrow R_{YY}(\tau) = \int_{-\infty}^{\infty} h(\tau_1)\, R_{YX}(\tau-\tau_1)\, d\tau_1$$

$$\boxed{\therefore R_{YY}(\tau) = h(\tau) * R_{YX}(\tau) = R_{YX}(\tau) * h(\tau)}$$

* Auto Correlation Function Response of linear System:

The autocorrelation function o/p response of systems is defined by

$$\text{ACF of } Y(t) = R_{YY}(t_1,t_2) = E[Y(t_1)\; Y(t_2)].$$

O/p response of linear system $Y(t) = h(t) * x(t)$

$$= \int h(\tau)\, x(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} h(\alpha)\, x(t-\alpha)\, d\alpha$$

$$Y(t_1) = h(t_1) * X(t_1) = \int_{-\infty}^{\infty} h(\tau)\, x(t_1-\tau)\, d\tau = \int_{-\infty}^{\infty} h(\tau_1)\, x(t_1-\tau_1)\, d\tau_1$$

$$Y(t_2) = h(t_2) * X(t_2) = \int_{-\infty}^{\infty} h(\tau)\, x(t_2-\tau)\, d\tau = \int_{-\infty}^{\infty} h(\tau_2)\, x(t_2-\tau_2)\, d\tau_2$$

$$R_{YY}(t_1,t_2) = E\left(\int_{-\infty}^{\infty} h(\tau_1)\, X(t_1-\tau_1)\, d\tau_1 \cdot \int_{-\infty}^{\infty} h(\tau_2)\, X(t_2-\tau_2)\, d\tau_2\right]$$

$$= E\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)\, h(\tau_2)\, x(t_1-\tau_1)\, X(t_2-\tau_2)\, d\tau_1\, d\tau_2\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)\, h(\tau_2)\, E\left[x(t_1-\tau_1)\; x(t_2-\tau_2)\right] d\tau_1\, d\tau_2$$

We know that if $x(t)$ is WSS random process then

$$R_{xx}(t_1,t_2) = E[x(t_1)\, x(t_2)] = R_{xx}(t_2-t_1)$$

$$E[x(t_1-\tau_1)\, x(t_2-\tau_2)] = R_{xx}((t_2-\tau_2)-(t_1-\tau_1)]$$

$$= R_{xx}((t_2-t_1)+\tau_1-\tau_2)$$

$$= R_{xx}((t_2-t_1)+\tau_1-\tau_2)$$

$$\Rightarrow R_{YY}(t_1,t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)R_{XX}(t_2-t_1+\tau_1-\tau_2)\,d\tau_1 d\tau_2$$

For WSS random process $t_2 - t_1 = \tau$

$$R_{YY}(t_1,t_2) = R_{YY}(t_2-t_1) = R_{YY}(\tau)$$

$$\therefore R_{YY}(t_1,t_2) = R_{YY}(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(\tau+\tau_1-\tau_2)\,h(\tau_1)h(\tau_2)\,d\tau_1 d\tau_2$$

$$\boxed{R_{YY}(\tau) = h(-\tau) * R_{XX}(\tau) * h(\tau) = R_{XX}(\tau) * h(\tau) * h(-\tau)}$$

From CCF properties (i) and (iii)

$$R_{XY}(\tau) = h(\tau) * R_{XX}(\tau)$$

$$R_{YY}(\tau) = h(-\tau) * R_{XY}(\tau)$$

then $R_{YY}(\tau) = h(-\tau) * \left(h(\tau) * R_{XX}(\tau)\right) = h(-\tau) * h(\tau) * R_{XX}(\tau)$

$$R_{YY}(\tau) = R_{XX}(\tau) * h(\tau) * h(-\tau)$$

$\therefore$ ACF of o/p response is a function of τ only not absolute time 't'. Hence, ACF of o/p response is also a WSS random process.

*** Power Spectrum Density of o/p response of LTI system,**

(OR)

Test

Derive relationship b/n PSD of input and output random process of linear system or LTI system.

Statement: If the PSD of input random process $x(t)$ is $S_{XX}(\omega)$ and $H(\omega)$ is transfer function of linear system, then the PSD of o/p response of linear system is given by

$$\boxed{S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)}$$ i.e.,

o/p PSD of $Y(t)$ = $|H(\omega)|^2$ i/p PSD of $X(t)$

Proof: The ACF of o/p response of LTI system is defined by $y(t) = R_{YY}(\tau) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} R_{XX}(\tau+\tau_1-\tau_2)\, h(\tau_1) h(\tau_2)\, d\tau_1\, d\tau_2$

Apply fourier transform on both sides, we get

$$F[R_{YY}(\tau)] = F\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(\tau+\tau_1-\tau_2) h(\tau_1) h(\tau_2)\, d\tau_1\, d\tau_2\right]$$

$$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(\tau+\tau_1-\tau_2)\, h(\tau_1)\, h(\tau_2)\, d\tau_1\, d\tau_2\right] e^{-j\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau_1) \int_{-\infty}^{\infty} h(\tau_2) \left[\int_{-\infty}^{\infty} R_{XX}(\tau+\tau_1-\tau_2)\, e^{-j\omega\tau}\, d\tau\right] d\tau_2\, d\tau_1$$

$$= \int_{-\infty}^{\infty} h(\tau_1)\, e^{j\omega\tau_1} \int_{-\infty}^{\infty} h(\tau_2)\, e^{-j\omega\tau_2} \int_{-\infty}^{\infty} R_{XX}(\tau+\tau_1-\tau_2)\, e^{-j\omega\tau} e^{-j\omega\tau_1} e^{j\omega\tau_2}\, d\tau\, d\tau_2\, d\tau_1$$

$$= \int_{-\infty}^{\infty} h(\tau_1)\, e^{j\omega\tau_1} \int_{-\infty}^{\infty} h(\tau_2)\, e^{-j\omega\tau_2} \left[\int_{-\infty}^{\infty} R_{XX}(\tau+\tau_1-\tau_2)\, e^{-j\omega(\tau+\tau_1-\tau_2)}\, d\tau\right] d\tau_2\, d\tau_1$$

$$= \int_{-\infty}^{\infty} h(\tau_1)\, e^{j\omega\tau_1} \int_{-\infty}^{\infty} h(\tau_2)\, e^{-j\omega\tau_2} \left[\int_{-\infty}^{\infty} R_{XX}(t)\, e^{-j\omega t}\, dt\right] d\tau_2\, d\tau_1$$

Put $\tau + \tau_1 - \tau_2 = t$

$d\tau = dt$

$\tau \to -\infty \Rightarrow t = -\infty$

$\tau \to \infty \Rightarrow t = \infty$

$$= \int_{-\infty}^{\infty} h(\tau_1)\, e^{j\omega\tau_1} \int_{-\infty}^{\infty} h(\tau_2)\, e^{-j\omega\tau_2}\, F[R_{XX}(t)]\, d\tau_2\, d\tau_1$$

$d\tau_2$

We know $R_{XX}(\tau) \overset{F.T}{\longleftrightarrow} S_{XX}(\omega)$

$$R_{YY}(\tau) \longleftrightarrow S_{YY}(\omega)$$

Now, $$S_{YY}(\omega) = \int_{-\infty}^{\infty} h(\tau_1) e^{j\omega\tau_1} \int_{-\infty}^{\infty} h(\tau_2) e^{-j\omega\tau_2} S_{XX}(\omega)\, d\tau_2\, d\tau_1$$

$\because$ independent operator

$$= S_{XX}(\omega) \left(\int_{-\infty}^{\infty} h(\tau_1) e^{j\omega\tau_1} d\tau_1 \right) \left(\int_{-\infty}^{\infty} h(\tau_2) e^{-j\omega\tau_2} d\tau_2 \right)$$

τ We know that $F[h(\tau)] = H(\omega) = \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau$

Replacing $\tau = \tau_2$

τ_2 $$F[h(\tau_2)] = H(\omega) = \int_{-\infty}^{\infty} h(\tau_2) e^{-j\omega\tau_2} d\tau_2$$

Replacing $\tau = \tau_1$

$\omega\tau_2$

$\frac{}{2} d\tau_1$ $$F[h(\tau_1)] = H(\omega) = \int_{-\infty}^{\infty} h(\tau_1) e^{-j\omega\tau_1} d\tau_1$$

$$H^*(\omega) = \int_{-\infty}^{\infty} h^*(\tau_1) \left(e^{-j\omega\tau_1}\right)^* d\tau_1$$

$\because$ Conjugation of real function is real.

$$H^*(\omega) = \int_{-\infty}^{\infty} h(\tau_1) e^{j\omega\tau_1} d\tau_1$$

Now, we have

$$S_{YY}(\omega) = S_{XX}(\omega)\; H^*(\omega)\; H(\omega)$$

$\left\{ \because x^*(t)\, x(t) = |x(t)|^2 \right.$

$$= S_{XX}(\omega)\, |H(\omega)|^2$$

$$\therefore S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$$

Hence proved.

* Cross Power Spectral Densities:

(i) $S_{XY}(\omega) = H(\omega) \cdot S_{XX}(\omega)$

Proof: We know $R_{XY}(\tau) = h(\tau) * R_{XX}(\tau)$ {From case (i) in CCF o/p response}

Applying fourier transform, we have

$$F[R_{XY}(\tau)] = F[h(\tau) * R_{XX}(\tau)]$$

We know that, $F[g_1(t) * g_2(t)] = F[g_1(t)] \cdot F[g_2(t)]$

$$= G_1(\omega) \cdot G_2(\omega)$$

$$h(\tau) \xleftrightarrow{F.T} H(\omega)$$

$$R_{XX}(\tau) \leftrightarrow S_{XX}(\omega)$$

$$R_{XY}(\tau) \leftrightarrow S_{XY}(\omega)$$

$$\Rightarrow F[R_{XY}(\tau)] = F[h(\tau)] \cdot F[R_{XX}(\tau)]$$

$$\boxed{\therefore S_{XY}(\omega) = H(\omega) \cdot S_{XX}(\omega)}$$

Hence proved.

(ii) $S_{YX}(\omega) = H^*(\omega) S_{XX}(\omega) = H(-\omega) S_{XX}(\omega)$

Proof: We know $R_{YX}(\tau) = h(-\tau) * R_{XX}(\tau)$ {from case (ii) in CCF o/p response}

Applying fourier transform, we have

$$F[R_{YX}(\tau)] = F[h(-\tau) * R_{YX}(\tau)]$$

We know that $F[g_1(t) * g_2(t)] = F[g_1(t)] \cdot F[g_2(t)]$

$$= G_1(\omega) \cdot G_2(\omega)$$

$$h(\tau) \xleftrightarrow{F.T} H(\omega) \Rightarrow h(-\tau) \longleftrightarrow H(-\omega) = H^*(\omega)$$

$$R_{YX}(\tau) \longleftrightarrow S_{YX}(\omega)$$

$$R_{XX}(\tau) \longleftrightarrow S_{XX}(\omega)$$

$$\Rightarrow F[R_{YX}(\tau)] = F[h(-\tau)] \cdot F[R_{XX}(\tau)]$$

$$\boxed{\therefore S_{YX}(\omega) = H^*(\omega)\, S_{XX}(\omega)}$$

Hence proved.

o/p ...

II method:

$$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$$

Proof. We know $R_{YY}(\tau) = h(-\tau) * h(\tau) * R_{XX}(\tau)$

{ From ACF o/p respon

Apply fourier transform on both sides, we get

$$F[R_{YY}(\tau)] = F[h(-\tau) * [h(\tau) * R_{XX}(\tau)]]$$

We have $F[g_1(\tau) * g_2(\tau)] = F[g_1(\tau)] \cdot F[g_2(\tau)]$

$$= G_1(\omega) \cdot G_2(\omega)$$

$$F[R_{YY}(\tau)] = F[h(-\tau)] \cdot F[h(\tau) * R_{XX}(\tau)]$$

Again by the property, we have

$$F[h(\tau) * R_{XX}(\tau)] = F[h(\tau)] \cdot F[R_{XX}(\tau)]$$

$$\Rightarrow F[R_{YY}(\tau)] = F[h(-\tau)] \cdot F[h(\tau)] \cdot F[R_{XX}(\tau)]$$

$$h(-\tau) \longleftrightarrow H^*(\omega)$$

$$h(\tau) \longleftrightarrow H(\omega)$$

$$R_{XX}(\tau) \longleftrightarrow S_{XX}(\omega)$$

$$R_{YY}(\tau) \longleftrightarrow S_{YY}(\omega)$$

$$\Rightarrow S_{YY}(\omega) = H^*(\omega)\, H(\omega)\, S_{XX}(\omega)$$

$$\therefore S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$$

where $S_{XX}(\omega) \Rightarrow$ PSD of i/p R.P. $X(t)$

$H(\omega) \Rightarrow$ Transfer function or frequency response of LTI system

$S_{YY}(\omega) \Rightarrow$ PSD of o/p R.P $Y(t)$

Hence proved.

*Problems:

1. Let a random process X(t) having PSD $S_{XX}(\omega) = \frac{3}{49+\omega^2}$ is applied to a network whose impulse response is $h(t) = t^2 \, Exp(-7t)u(t)$ and o/p response is represented by Y(t).

(i) Find average power of i/p random process X(t).

(ii) Find PSD of o/p random process Y(t).

(iii) Find average power of o/p random process Y(t).

Sol: Given $S_{XX}(\omega) = \frac{3}{49+\omega^2}$

Impulse response of the given n/w $= h(t) = t^2 \exp(-7t) u(t)$

(i) Average power of i/p random process $X(t) = P_{XX}$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{3}{49+\omega^2}\right) d\omega$$

$$= \frac{1}{2\pi}\cdot 3\int_{-\infty}^{\infty}\left(\frac{1}{49+\omega^2}\right) d\omega$$

$$= \frac{1}{2\pi}\cdot 3\int_{-\infty}^{\infty}\left(\frac{1}{\omega^2+7^2}\right) d\omega$$

$$= \frac{3}{2\pi}\left[\frac{1}{7}\tan^{-1}\left(\frac{\omega}{7}\right)\right]_{-\infty}^{\infty} \quad \because \int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

$$= \frac{3}{14\pi}\left[\tan^{-1}\left(\frac{\omega}{7}\right)\right]_{-\infty}^{\infty}$$

$$= \frac{3}{14\pi}\left[\tan^{-1}\left(\frac{\infty}{7}\right) - \tan^{-1}\left(\frac{-\infty}{7}\right)\right]$$

$$= \frac{3}{14\pi}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right]$$

$$= \frac{3}{14\pi}\left[\frac{2\pi}{2}\right]$$

$$\boxed{\therefore P_{XX} = \frac{3}{14}\,W = 0.214 \text{ Watts}}$$

(ii) PSD of o/p random process $Y(t)$:

We know relationship b/w input, output PSDs of random process i.e,

Output PSD of $Y(t) = S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$

The given impulse response $h(t) = t^2 \cdot e^{-7t} u(t)$

We know $t^n e^{-at} u(t) \xleftrightarrow{F.T} \dfrac{n!}{(a+j\omega)^{n+1}}$

Here $n=2$, $a=7$ then $t^2 e^{-7t} u(t) \longleftrightarrow \dfrac{2!}{(7+j\omega)^{2+1}}$

$$= \frac{2}{(7+j\omega)^3}$$

and $h(t) \longleftrightarrow H(\omega)$

$$F[h(t)] = H(\omega) = F[t^2 e^{-7t} u(t)] = \frac{2}{(7+j\omega)^3}$$

$$|H(\omega)| = \left|\frac{2}{(7+j\omega)^3}\right|$$

$$= \frac{2}{|(7+j\omega)|^3}$$

$$= \frac{2}{\sqrt{(7^2+\omega^2)^3}}$$

$$|H(\omega)| = \frac{2}{(7^2+\omega^2)^{3/2}} \Rightarrow |H(\omega)|^2 = \left(\frac{2}{(7^2+\omega^2)^{3/2}}\right)^2$$

$$= \frac{4}{(7^2+\omega^2)^3}$$

$$\therefore |H(\omega)|^2 = \frac{4}{(49+\omega^2)^3}$$

Now $S_{YY}(\omega) = \left[\dfrac{4}{(49+\omega^2)^3}\right]\left[S_{XX}(\omega)\right]$

$$= \left[\frac{4}{(49+\omega^2)^3}\right]\left[\frac{3}{(49+\omega^2)}\right]$$

$$= \frac{12}{(49+\omega^2)^4}$$

$$\boxed{\therefore S_{YY}(\omega) = \frac{12}{(7^2+\omega^2)^4}}$$

(iii) Average power of o/p random process $Y(t)$ is

$$P_{YY} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YY}(\omega)\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{12}{(7^2+\omega^2)^4}\, d\omega$$

$$= \frac{6}{\pi}\int_{-\infty}^{\infty} \frac{1}{(7^2+\omega^2)^4}\, d\omega \qquad \because \int \frac{1}{(a^2+\omega^2)^4}\,dx = \frac{5\pi}{16a^7}$$

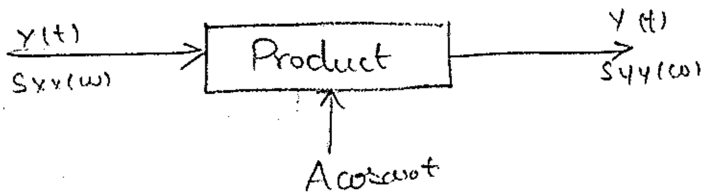
$$= \frac{\overset{3}{\cancel{6}}}{\cancel{\pi}}\ \frac{5\cancel{\pi}}{\underset{8}{\cancel{16}}\times(7)^7}$$

$$= \frac{15}{8\times(7)^7}$$

$$= 2\cdot 27\times 10^{-6}\ W$$

$$\boxed{\therefore P_{YY} = 2\cdot 27\ \mu W}$$

2. Let $X(t)$ is input random process, find $R_{YY}(\tau)$ and $S_{YY}(\omega)$ in terms of i/p PSD $S_{XX}(\omega)$ for the product device n/w shown in figure. ~~Find~~



Sol: Output random process $= Y(t) =$ output of product device

$$= X(t)\, A\cos\omega_0 t = A\, X(t) \cos\omega_0 t$$

The ACF of o/p random process $Y(t) = R_{YY}(\tau)$

$$= A\left[E[Y(t)\ Y(t+\tau)]\right]$$

$$E[Y(t)\ Y(t+\tau)] = E\left[A\ X(t)\cos\omega_0 t \cdot A\ X(t+\tau)\cos(\omega_0 t+\tau)\right]$$

$$= E\left[A^2 X(t)\ X(t+\tau)\cos\omega_0 t\cos(\omega_0 t+\omega_0\tau)\right]$$

$$= A^2 E[X(t)\ X(t+\tau)]\cos\omega_0 t\cos(\omega_0 t+\omega_0\tau)$$

We know $\Rightarrow$ $R_{XX}(\tau) = E[X(t)\ X(t+\tau)]$ for WSS

$$\cos A\cos B = \frac{\cos(A-B)+\cos(A+B)}{2}$$

$$= A^2 \cdot R_{XX}(\tau)\left[\frac{\cos(\omega_0 t-\omega_0 t-\omega_0\tau)+\cos(\omega_0 t+\omega_0 t+\omega_0\tau)}{2}\right]$$

$$= A^2 R_{XX}(\tau)\left[\frac{\cos(-\omega_0\tau)+\cos(2\omega_0 t+\omega_0\tau)}{2}\right]$$

$\because \cos(-\theta) = \cos\theta$

$$= A^2 R_{XX}(\tau)\left[\frac{\cos\omega_0\tau}{2}+\frac{\cos(2\omega_0 t+\omega_0\tau)}{2}\right]$$

$$= A^2 R_{XX}(\tau)\frac{\cos\omega_0\tau}{2} + \frac{A^2 R_{XX}(\tau)\cos(2\omega_0 t+\omega_0\tau)}{2}$$

$$E[Y(t)Y(t+\tau)] = \frac{A^2}{2}R_{XX}(\tau)\cos\omega_0\tau + \frac{A^2}{2}R_{XX}(\tau)\cos(2\omega_0 t+\omega_0\tau)$$

Now $R_{YY}(\tau) = A\left[E(Y(t)\ Y(t+\tau))\right]$

$$R_{YY}(\tau) = A\left[\frac{A^2}{2}R_{XX}(\tau)\cos(\omega_0\tau) + \frac{A^2}{2}R_{XX}(\tau)\cos(2\omega_0 t+\omega_0\tau)\right]$$

$$= A\left[\frac{A^2}{2}R_{XX}(\tau)\cos\omega_0\tau\right] + A\left[\frac{A^2}{2}R_{XX}(\tau)\cos(2\omega_0 t+\omega_0\tau)\right]$$

$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left[\frac{A^2}{2}R_{XX}\cos(\omega_0\tau)\right]dt + \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left[\frac{A^2}{2}R_{XX}(\tau)\cos(2\omega_0 t+\omega_0\tau)\right]dt$$

$$= \frac{A^2}{2} R_{xx}(\tau)\cos(\omega_0\tau) \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T} 1\, dt + \frac{A^2}{2} R_{xx}(\tau) \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T} \cos(2\omega_0 t + \omega_0\tau)\, dt$$

$$= \frac{A^2}{2} R_{xx}(\tau)\cos\omega_0\tau \underset{T\to\infty}{Lt} \frac{1}{2T}\left[t\right]_{-T}^{T} + \frac{A^2}{2} R_{xx}(\tau) \underset{T\to\infty}{Lt} \frac{1}{2T}(0)$$

$$\because \int_{-T}^{T} \cos(n\omega_0 t + \theta)\, dt = 0$$

$$= \frac{A^2}{2} R_{xx}(\tau)\cos(\omega_0\tau) \underset{T\to\infty}{Lt} \frac{1}{2T}\left[T-(-T)\right] + 0$$

$$\boxed{\therefore R_{YY}(\tau) = \frac{A^2}{2} R_{xx}(\tau)\cos(\omega_0\tau)}$$

Output PSD $= S_{YY}(\omega) = F[R_{YY}(\tau)]$

$$= F\left[\frac{A^2}{2} R_{xx}(\tau)\cos(\omega_0\tau)\right]$$

$$\because F[a\, g(\tau)] = a\, F[g(\tau)]$$

$$= \frac{A^2}{2} F\left[R_{xx}(\tau)\cos(\omega_0\tau)\right]$$

We know modulation theorem

$$g(\tau) \xleftrightarrow{F.T.} G(\omega) \text{ then}$$

$$g(\tau)\cos(\omega_0\tau) \longrightarrow \frac{1}{2}\left[G(\omega-\omega_0) + G(\omega+\omega_0)\right]$$

$$R_{xx}(\tau) \longleftrightarrow S_{xx}(\omega)$$

$$R_{xx}(\tau)\cos(\omega_0\tau) \longleftrightarrow \frac{1}{2}\left[S_{xx}(\omega-\omega_0) + S_{xx}(\omega+\omega_0)\right]$$

$$\Rightarrow S_{YY}(\omega) = \frac{A^2}{2}\times\frac{1}{2}\left[S_{xx}(\omega-\omega_0) + S_{xx}(\omega+\omega_0)\right]$$

$$\boxed{\therefore S_{YY}(\omega) = \frac{A^2}{4}\left[S_{xx}(\omega-\omega_0) + S_{xx}(\omega+\omega_0)\right]}$$

3. Find transfer function, impulse response of o/p PSD, o/p average power, o/p ACF, i/p ACF, and i/p average power for the RC low pass filter network when applied i/p having PSD is Gaussian White noise PSD i.e $\frac{N_0}{2}$ and also find noise bandwidth of RC low pass filter.

Sol: A RC low pass filter is as shown in figure.

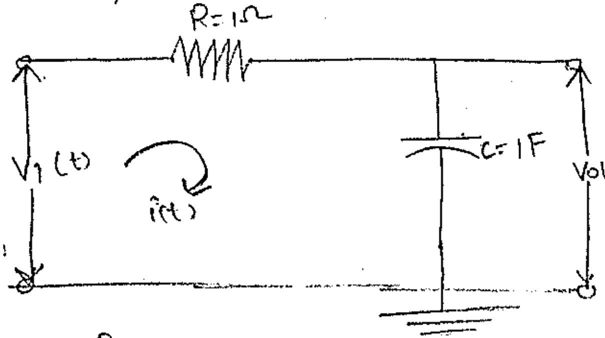


fig a: RC-Low pass filter

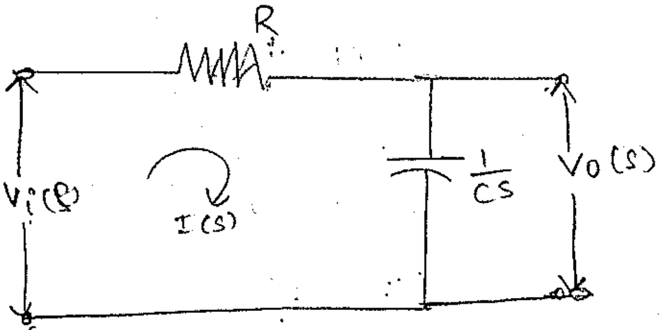


fig b: S-domain equivalent n/w

$$\text{i/p voltage} = V_i(s) = R\cdot I(s) + \frac{1}{Cs} I(s) = \left(R + \frac{1}{Cs}\right) I(s)$$

$$= \left[\frac{1+RCS}{CS}\right] I(s)$$

$$\text{o/p voltage} = V_o(s) = I(s)\cdot \frac{1}{CS}$$

$$\text{Transfer function in S-domain} = H(s) = \frac{V_o(s)}{V_i(s)}$$

$$= \frac{\cancel{I(s)}\,\frac{1}{\cancel{C}S}}{\left[\frac{1+RCS}{\cancel{C}S}\right]\cdot \cancel{I(s)}}$$

$$= \frac{1}{1+RCS}$$

$$\therefore H(s) = \frac{1}{RC}\left[\frac{1}{\frac{1}{RC}+s}\right]$$

Transfer function in frequency domain (or) frequency response $= H(\omega)$

$$= H(j\omega) = H(s)\Big|_{j\omega=s}$$

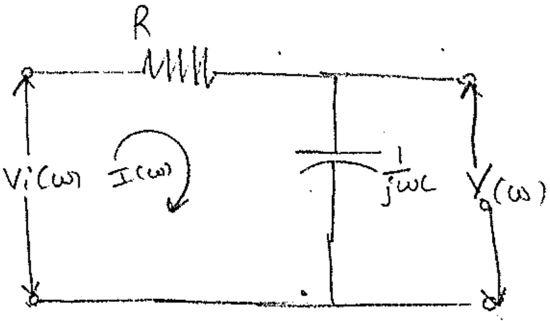


fig (c): frequency domain n/w

$$= \frac{1}{RC}\left[\frac{1}{\frac{1}{RC} + j\omega}\right]$$

$$\therefore H(\omega) = \frac{1}{1+jRC\omega}$$

Magnitude Response $|H(\omega)| = \left|\frac{1}{1+j\omega RC}\right|$

$$= \frac{1}{\sqrt{1+(\omega RC)^2}}$$

$$= \frac{1}{(RC)^2}\left[\frac{1}{\sqrt{\left(\frac{1}{RC}\right)^2 + \omega^2}}\right]$$

$$|H(\omega)|^2 = \frac{1}{(RC)^2}\left[\frac{1}{\sqrt{\left(\frac{1}{RC}\right)^2 + \omega^2}}\right]$$

$$= \frac{1}{RC^2}\,\frac{1}{\left(\frac{1}{RC}\right)^2 + \omega^2}$$

$$|H(\omega)|^2 = \frac{1}{1+\omega^2R^2C^2}$$

Phase response $= \angle H(\omega) = -\tan^{-1}\left(\frac{\omega RC}{1}\right)$

$$\angle H(\omega) = -\tan^{-1}(\omega RC)\left\{\because \frac{1}{a+jb} = -\tan^{-1}\left(\frac{b}{a}\right)\right\}$$

O/p PSD :

We know $S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$

Given i/p PSD $= S_{XX}(\omega) = \frac{N_0}{2}$ W/Hz

$$S_{YY}(\omega) = \frac{1}{1+\omega^2R^2C^2} \cdot \frac{N_0}{2}$$

$$\boxed{S_{YY}(\omega) = \frac{N_0}{2(1+\omega^2R^2C^2)} = \frac{N_0}{2R^2C^2((1/RC)^2+\omega^2)}}$$

O/P Average Power:

$$P_{YY} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YY}(\omega)\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{N_0}{2(1+\omega^2R^2C^2)}\, d\omega$$

$$= \frac{N_0}{2\pi \times 2}\int_{-\infty}^{\infty} \frac{1}{(1+\omega^2R^2C^2)}\, d\omega$$

$$= \frac{N_0}{4\pi}\int_{-\infty}^{\infty} \frac{1}{R^2C^2((1/RC)^2+\omega^2)}\, d\omega$$

$$= \frac{N_0}{4\pi R^2C^2}\int_{-\infty}^{\infty} \frac{1}{((1/RC)^2+\omega^2)}\, d\omega$$

$$= \frac{N_0}{4\pi \cancel{R^2C^2}} \cdot \frac{1}{\cancel{(1/RC)}} \tan^{-1}\left(\frac{\omega}{1/RC}\right)$$

$$= \frac{N_0}{4\pi}\left[\tan^{-1}(\omega RC)\right]_{-\infty}^{\infty}$$

$$= \frac{N_0}{4\pi}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right]$$

$$= \frac{N_0}{4\pi} \cdot [\pi]$$

$$\boxed{P_{YY} = \frac{N_0}{4RC}\ W}$$

O/p ACF:

$$R_{YY}(\tau) = F^{-1}\left[S_{YY}(\omega)\right]$$

$$= F^{-1}\left[\frac{N_0}{2R^2C^2\left(\frac{1}{(RC)^2}+\omega^2\right)}\right]$$

$$= \frac{N_0}{2R^2C^2} \; F^{-1}\left[\frac{1}{\left(\left(\frac{1}{RC}\right)^2+\omega^2\right)}\right]$$

We know $e^{-a|\tau|} \xleftrightarrow{F.T} \dfrac{2a}{a^2+\omega^2}$

$$\frac{1}{2a} e^{-a|\tau|} \longleftrightarrow \frac{1}{a^2+\omega^2}$$

If $a = \frac{1}{RC} \Rightarrow \dfrac{1}{2\left(\frac{1}{RC}\right)} e^{-\frac{1}{RC}|\tau|} \longleftrightarrow \dfrac{1}{(1/RC)^2+\omega^2}$

$$F^{-1}\left[\frac{1}{\left(\frac{1}{RC}\right)^2+\omega^2}\right] = \frac{RC}{2} e^{-\frac{|\tau|}{RC}}$$

$$\therefore R_{YY}(\tau) = \frac{N_0}{2R^{\cancel{2}}C^{\cancel{2}}} \cdot \frac{\cancel{R}\cancel{C}}{2} e^{-\frac{1}{RC}|\tau|} = \frac{N_0}{4RC} e^{-\frac{1}{RC}|\tau|}$$

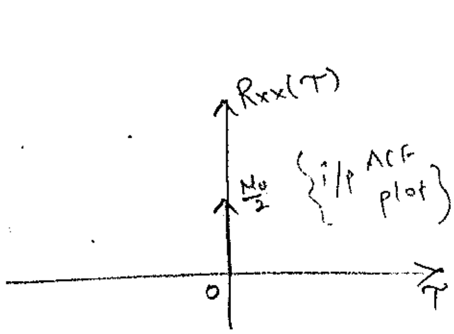
$$\boxed{R_{YY}(\tau) = \frac{N_0}{4RC} e^{-\frac{1}{RC}|\tau|}}$$

i/p ACF:

$$R_{XX}(\tau) = F^{-1}\left[S_{XX}(\omega)\right]$$

Given $S_{XX}(\omega) = \dfrac{N_0}{2}$

$$R_{XX}(\tau) = F^{-1}\left[\frac{N_0}{2}\right]$$



$$= \frac{N_0}{2} F^{-1}[1]$$

We know $\delta(\tau) \longleftrightarrow 1 \Rightarrow F^{-1}[1] \leftrightarrow \delta(\tau)$

$$\boxed{R_{XX}(\tau) = \frac{N_0}{2} \cdot \delta(\tau)} \text{ i.e. } \begin{cases} \frac{N_0}{2} & ; \tau = 0 \\ 0 & ; \tau \neq 0 \end{cases}$$

i/p Average Power:

$$P_{XX} = R_{XX}(0) = \frac{N_o}{2}\,\delta(0)$$

$$\boxed{\therefore P_{XX} = \frac{N_o}{2} \text{ watts}}$$

Noise bandwidth:

The noise bandwidth of network is defined by

$$W_N = \frac{\int_0^\infty |H(\omega)|^2\, d\omega}{|H(0)|^2}$$

$$\int_0^\infty |H(\omega)|^2\, d\omega = \int_0^\infty \frac{1}{(RC)^2}\left(\frac{1}{\left(\frac{1}{RC}\right)^2 + \omega^2}\right) d\omega$$

$$= \frac{1}{(RC)^2}\left[\frac{1}{1/RC}\tan^{-1}\left(\frac{\omega}{(1/RC)}\right)\right]_0^\infty$$

$$\cancel{= \frac{RC}{R^2C^2}\tan^{-1}\left(\frac{\pi}{2} - 0\right)}$$

$$= \frac{RC}{R^2C^2}\left[\tan^{-1}\omega RC\right]_0^\infty$$

$$= \frac{1}{R.C}\left(\frac{\pi}{2} - 0\right)$$

$$= \frac{\pi}{2RC}$$

$$|H(0)|^2 = \left(\frac{1}{RC}\right)^2\left[\frac{1}{(1/RC)^2 + 0^2}\right] = 1$$

$$\boxed{W_N = \pi/2RC = \pi/2RC \text{ rad/sec}}$$

Cutoff frequency $= W_c = \frac{1}{RC}$ rad/sec

$$\boxed{W_N = \frac{\pi}{2} W_c}$$

$$\boxed{B_N = \frac{\omega_N}{2\pi}}$$

$$\omega_N = 2\pi B_N \Rightarrow \quad B_N = \frac{\omega_N}{2\pi}\ Hz$$

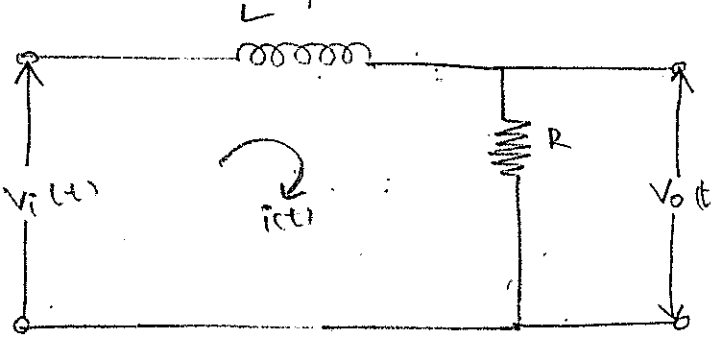
$$= \frac{\frac{\pi}{2RC}}{2\pi}$$

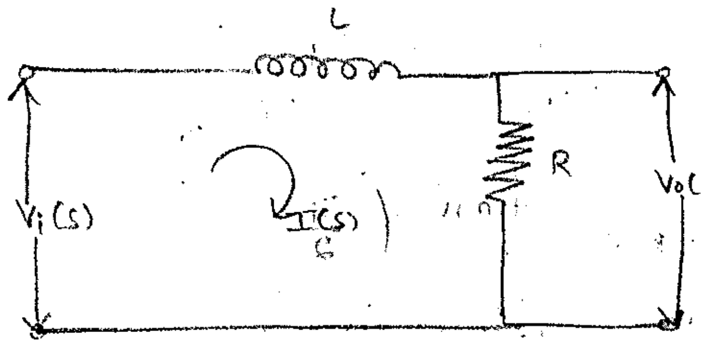
$$\boxed{B_N = \frac{\pi}{4\pi RC}\ Hz}$$

4. Repeat the above problem for the given RL n/w.

$H(S) = \frac{R}{R+SL}$; $H(j\omega) = \frac{R}{R+j\omega L}$

$h(t) = \frac{R}{L} e^{-\frac{R T}{L}} u(T)$





i/p voltage $= V_i(s) = SLI(s) + R[I(s)] = I(s)(R+L)$

O/p voltage $= V_o(s) = I(s) \cdot R = R\,I(s)$

Transfer function in S-domain $= H(s) = \frac{V_o(s)}{V_i(s)}$

$$= \frac{R(\cancel{I(s)})}{\cancel{I(s)}(R+j\omega L)}$$

$$\therefore H(s) = \frac{R}{R+j\omega L}$$

(OR)

$$H(s) = \frac{V_o(s)}{V_i(s)}$$

$$\boxed{\therefore H(s) = \frac{R}{R+SL}}$$

$$= \frac{\sqrt{\pi}}{2}\, G\left(\frac{\sqrt{\pi}\,\omega}{2}\right)$$

$$= \frac{\sqrt{\pi}}{2}\, e^{-\left(\frac{\left(\frac{\sqrt{\pi}\,\omega}{2}\right)^2}{4\pi}\right)} = \frac{\sqrt{\pi}}{2}\, e^{-\frac{\omega^2}{16}}$$

$$= \frac{\sqrt{\pi}}{2}\, e^{-\frac{\pi^2 f^2}{4}}$$

$$\therefore\ e^{-4\tau^2} \xleftrightarrow{F.T} \frac{\sqrt{\pi}}{2}\, e^{-\omega^2/16} = \frac{\sqrt{\pi}}{2}\, e^{-\frac{\pi^2 f^2}{4}}$$

$(\because \omega = 2\pi f)$

$$\boxed{\therefore S_x(\omega) = 6\pi\delta(\omega) + 2 \times \frac{\sqrt{\pi}}{2} \cdot e^{-\omega^2/16}}$$

(b) Average power in $X(t)$:

We know ACF of $X(t) = R_X(\tau) = 3 + 2e^{-4\tau^2}$

$$P_X = R_X(0) = 3 + 2e^{-4(0)^2} = 3 + 2 = 5$$

$$\boxed{P_X = 5\,W}$$

(c)
$$P_X = \frac{1}{2\pi}\int_{-1/\sqrt{2}}^{1/\sqrt{2}} S_X(\omega)\, d\omega$$

$$= \frac{1}{2\pi}\int_{-1/\sqrt{2}}^{1/\sqrt{2}} \left(6\pi\delta(\omega) + \sqrt{\pi}\, e^{-\omega^2/16}\right) d\omega$$

$$= \frac{1}{2\pi} \times 6\pi \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \delta(\omega)\, d\omega + \frac{1}{2\pi} \cdot \sqrt{\pi}\int_{-1/\sqrt{2}}^{1/\sqrt{2}} e^{-\frac{\omega^2}{16}}\, d\omega$$

$$= 3 \times \delta(\omega)\Big|_{\omega=0} + \frac{1}{2\pi} \times \sqrt{\pi} \times 2\int_0^{1/\sqrt{2}} e^{-\frac{\omega^2}{16}}\, d\omega = 3 \times 1 + \frac{1}{\sqrt{\pi}}\int_0^{1/\sqrt{2}} e^{-\left(\frac{\omega}{4}\right)^2} d\omega$$

Put $\frac{\omega}{4} = \tau \Rightarrow d\omega = 4\,d\tau$

$\omega \to 0 : \tau \to 0$

$\omega = 1/\sqrt{2} : \tau \to \frac{1}{4\sqrt{2}}$

$$= 3 + \frac{1}{\sqrt{\pi}} \int_0^{1/4\sqrt{2}} e^{-\tau^2} d\tau$$

$$= 3 + \frac{4}{\sqrt{\pi}} \phi\left(\frac{1}{4\sqrt{2}}\right)$$

$$\phi(x) = \int_0^x e^{-\tau^2} d\tau .$$

$$\tau \to \frac{1}{4\sqrt{2}} \to 0$$